



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

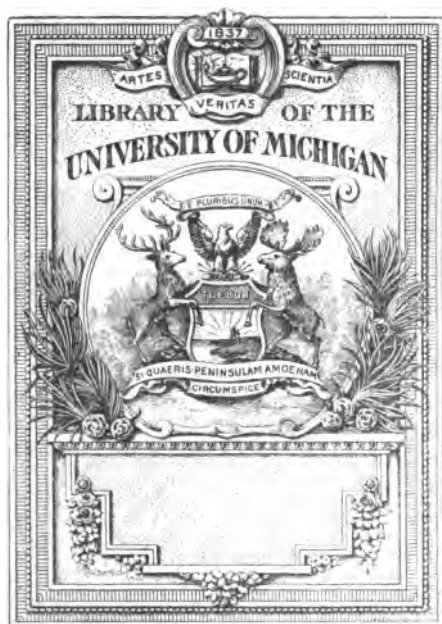
Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>



MATHEMATICS

0A

39

.M22

3/2

1

A SCHOOL COURSE OF MATHEMATICS

BY

DAVID MAIR

•

OXFORD
AT THE CLARENDON PRESS
1907

HENRY FROWDE, M.A.
PUBLISHER TO THE UNIVERSITY OF OXFORD
LONDON, EDINBURGH
NEW YORK AND TORONTO

©2004. 14. 9. 3.

PREFACE

THE material for mathematical education may be chosen for the value of the knowledge of the material itself, or for the value of the training got in the course of acquiring the knowledge. The knowledge-value is greater the more closely the thing studied is related to human life and interests, and less the more remote it is from these. As to training-value, the appropriate progress from concrete to abstract is most possible in connexion with things of concrete human interest; so that a selection of material made for its knowledge-value is confirmed by the criterion of training-value.

In range this book is in general agreement with the practice of our schools; containing the Geometry, Algebra, and Trigonometry usually read by pupils that do not specialize in mathematics. In detail there are a few differences. Thus the addition theorem in Trigonometry is of no great use for the solution of triangles, nor has the manipulation involved sufficient training-value to justify the inclusion of the theorem. On the other hand, a certain amount of Solid Geometry is included from a belief in its value for both knowledge and training.

The more the pupils can develop the subject for themselves and without help from the teacher, the truer is their knowledge of it and the more valuable the training received in the process; but care must be taken that this intensive method does not too greatly restrict the range of knowledge. This book has the form of a summarized discussion between teacher and pupils. In the earlier chapters the discussion is given in some detail. In later chapters it is more condensed, so that towards the end of the book the reply put in the pupil's mouth is the final formal conclusion,

which is reached only after a good deal of discussion. The order of the development of the subject must depend to some extent on the suggestions made by the pupils in discussion, and so must vary from class to class. This book shows one possible development; more importance is attached to the method than to the order of development. In order that the discussion may be a real one, each discussion should take place before the pupils look at the corresponding part of the book.

Some teachers may for the sake of drill and mechanical dexterity desire an increase in the number of exercises; they can easily multiply them by numerical alteration of the data. But I venture to think that this country pays too much attention to dexterity, and that to work honestly through the problems of the text and the exercises will result in as great a degree of dexterity as any one should require.

Occasional reference is made to principles. For a discussion of principles readers are referred to *Some Principles of Mathematical Education* by Mr. Benchara Branford, which is being issued by the Clarendon Press. The writing of my book has been in part due to reading Mr. Branford's manuscript.

Valuable help in suggestions and proof reading has been given by Messrs. Leonard Blaikie, Benchara Branford, J. G. Hamilton, and E. L. Kearney.

Most of the exercises are taken from examination papers of the Civil Service Commission, by the permission of the Controller of His Majesty's Stationery Office.

Criticisms and suggestions will be gratefully received. I shall be especially glad to hear of pupils' suggestions which (whether right or wrong) have proved fruitful for the development of the subject.

DAVID MAIR.

BANSTEAD, SURREY,
November, 1906.

CONTENTS

	PAGE
CHAPTER I. THE CACHE	1
Determination of position of cache. Points of the compass; right angle; degree; circle. Conclusions from experiment and from general reasoning. Isosceles triangle. Chord properties of circle. Work is to be done by pupils, not by teacher. To bisect a line; to draw a perpendicular. Converse propositions.	
CHAPTER II. POSITION OF CHAIR ON FLOOR	17
Position of a point given by distances from walls. Parallels. Loci. Symbols, and their right use.	
CHAPTER III. AREAS	27
Meaning of word 'Geometry'. Units of area. Use of brackets; precedence of $\times$ over $+$. Suitable degree of approximation. Area of rectangle, of right-angled triangle, of any triangle, of a curvilinear figure. Graphs. Algebraic expressions.	
CHAPTER IV. VOLUMES	45
Air-space of a room. Volume of a cylinder (gallon measure). Circumference of circle $\div$ diameter. Area of circle $\div$ square on radius. Averages. Volume of reservoir. Money lent at interest; place and value of abbreviations.	
CHAPTER V. TO COPY A MAP, SIZE UNCHANGED	59
Copy made by measurement of distances; number of measurements needed. Case of three places, other methods. Plane table. To make an angle equal to a given angle; to bisect an angle. Conditions that fix a triangle. Variation of angles of a triangle that has two sides given. Triangle fixed by three sides, by two sides and an angle, by a side and two angles; corresponding congruences. Slider crank pair.	

	PAGE
CHAPTER VI. FURTHER MENSURATION	83
To find the dimensions of a standard gallon measure. Comparison of Fahrenheit and Centigrade thermometers; simple equations. Money and debt; negative numbers. Sheep enclosure; quadratic equations.	
CHAPTER VII. ON CRANE PROBLEMS	98
Pupil's rate of advance. Wall crane; surface; circle; plane. Shear-legs; degrees of freedom; conditions that fix position of load. Traveller crane; parallels; per- pendicular to a plane. Scotch crane. Derrick crane. Co-ordinates. To make a plane surface.	
CHAPTER VIII. ANGLE-SUM PROPERTY. PARALLELS AND AREAS	122
Various approaches to the fact that the three angles of a triangle together make 180°. Properties of parallels. Construction of parallels; ruler and compasses, set-square, T-square. Parallel rulers; properties of parallelogram. Area of parallelogram, triangle, polygon; to make triangle or square equal in area to any polygon.	
CHAPTER IX. ON DRAWING TO SCALE	144
Plans of playground to various scales by chain-surveying; comparison of plans with one another and with play- ground. Plans by means of plane table; comparison; meaning of ratio. Horse-dealing; index notation; property of indices. Comparison of ratios involving incommen- surable numbers. Conditions that fix the shape of a triangle; sine of an angle. Areas of similar figures; volumes of similar solids.	
CHAPTER X. ON SIGNALLING	171
Morse Code; summation of series. Semaphore signalling. Flag signalling. Rosettes; combinations of n things p at a time. Binomial theorem. Interest on a loan.	
CHAPTER XI. ON CARRYING WATER	187
Perpendicular and slant distances of a point from a line. The greater side of a triangle is opposite the greater angle; and converse. Each bisector of the vertical angle of a tri- angle divides the base in the ratio of the sides. Calculation of the figure; trigonometrical expressions.	

CONTENTS

vii

	PAGE
CHAPTER XII. ON SHORTENING CALCULATIONS.	
SLIDE RULE	199
Table of powers of 2 used to avoid multiplication ; powers of 1·01 more useful. Table of powers replaced by graph. Graph replaced by graduated line ; slide rule. Laws of indices ; zero and negative indices ; square root from graph $y=1·01^x$; index $1/2$.	
CHAPTER XIII. DANGER ANGLE. PROPERTIES OF CIRCLES	216
Danger angle ; the angle in a segment of a circle is constant ; angles greater than 180° . Tangent to a circle as limit of line cutting circle. Relations among angles of cyclic quadrilateral ; limit when two vertices coalesce. Rectangle properties of a circle ; to draw a tangent.	
CHAPTER XIV. LOGARITHMS	236
Index-power table with 1·001 as base ; advantages of base 1·0028. Negative and fractional indices. Logarithms with base 10 ; checking table of logarithms. Calculation with logarithms ; root extraction ; laws of logarithms.	
CHAPTER XV. THEOREM OF PYTHAGORAS.	255
Theorem arrived at in various ways ; converse. Determination of a right-angled triangle from various data. Quadratic equations. Calculation of right-angled triangle. Conditions that fix shape of right-angled triangle. To find right-angled triangle with given area and perimeter ; simultaneous quadratic equations.	
CHAPTER XVI. CALCULATION OF A TRIANGLE	278
Calculation from $a \ A \ B$; trigonometrical relations ; case of $\angle B$ obtuse ; sine and cosine of an obtuse angle. Calculation from $a \ b \ B$; various cases. Calculation from $b \ c \ A$; tangent of obtuse angle. Calculation from $a \ b \ c$; case of obtuse angled triangle ; projection on a line. Extension of theorem of Pythagoras. Area of a triangle. A quadratic equation. Calculation of a triangle from the co-ordinates of the vertices ; and from other data.	
CHAPTER XVII. SOLID GEOMETRY	299
Existence of perpendicular to a plane. Perpendiculars to a plane are parallel ; converse. Number of points that fix a plane. Existence of parallel planes. Angle between	

two planes. Position of electric lamp fixed by lengths of three supporting strings; discussion of the figure by drawing and by calculation. Volume of a pyramid. Position of solid body fixed by six conditions. Conditions that fix a framework in form and in position. Congruence of solids. Images. Similarity. Circle and sphere. Volumes of similar solids.	PAGE
TABLE OF WEIGHTS AND MEASURES	371
ANSWERS TO CERTAIN NUMERICAL QUESTIONS	372
INDEX TO EXPLANATIONS OF TERMS AND PROPERTIES OF FIGURES	376

A SCHOOL COURSE OF MATHEMATICS

CHAPTER I

THE CACHE

1. A boy sometimes makes in a garden a cache or hiding-place for treasures, and covers it over so as to be indistinguishable. By what measurements can he fix the spot so as to find it again?

The pupils should be asked for suggestions, and their proposals discussed till a possible method is found. If the discussion can take place in a garden, it will go better. They will suggest measuring its distance from trees, posts, or other known points. Suppose the cache C is 7 feet from an apple-tree A , 12 feet from a plum-tree P , 4 feet from a cherry-tree T , and so on.

2. The cache is 7 feet from the tree A . Mark all the points at this distance from A ; in the garden if possible, with a piece of string, or, failing that, on paper with string or compasses, using 1 centimetre to represent 1 foot. These points make a curve that is called a circle, the apple-tree is at the centre of the circle, and the distance of every point of the circle from the apple-tree (7 feet) is called the radius of the circle. The cache lies somewhere on this circle. Now take the next measurement, 12 feet from the plum-tree P , and mark all the points that satisfy this condition, that is, all the points that are 12 feet from the plum-tree. What kind of curve does this give? What is its centre? and its radius? The cache lies on this line also.

From these two measurements what do you know of the position of the cache?—It must be at one of the points where these two curves cross, and we could find it by digging at these points.

But how could you avoid digging at the wrong point?—We could use another measurement.

Any other way?—We could note that the

cache lies north or east of the line joining the apple- and plum-trees, or note in some other way on which side of the line it lies.

3. Suppose the trees A and P to be 9 feet apart, and the cache 11 feet from A and 7 feet from P . Mark the trees on paper, 1 cm. representing 1 foot, and mark the two possible positions of the cache. With the same two trees, and the cache supposed to be 10 feet from A and 8 feet from P , mark its possible positions.

Suppose that the boy noted the position by fastening one end of a string at A , stretching it to C and making a knot there and fastening the string there, and stretching it from there to P , and cutting it off at P . How could he use this string to find the cache? Do it, taking the trees 9 feet apart and the two parts of the string 11 and 7 feet long. How many points might he find if he forgot whether 11 feet was the distance from A or from P ? Mark them all.

Suppose that when he tried to use the string he found the knot had slipped, what would he know of the position of the cache? Use the same lengths as before, and mark all the positions he would get by supposing the knot at different points along the string. Use the string or your compasses, whichever you like.

4. Are there any positions of the knot on the string that will not do?—Clearly 1 foot from an end will not do.

How does it fail? Distinguish between the positions that will do and the positions that won't. (The pupils will find by experiment that the knot must be at least 4.5 feet from an end.)

The two distances from the trees have been supposed to make up 18 feet, or in the drawing 18 cm., and we have found that the distance from each tree must be at least 4.5 feet, or in the drawing 4.5 cm. Suppose this restriction that the two distances make up 18 feet or cm. removed, and find by experiment what relation there must be between the two distances of the cache from the trees, the trees being 9 cm. apart on your drawing. Take 3 cm. as the distance from A , and take in succession 3, 4, 5, 6, . . . cm. as the distance from P . Then take 4 cm. as the distance from A , and the same succession of distances from

P ; and so on, till you see what the relation is, and till you can prove it without any drawing at all.—The sum of the distances must be greater than 9 cm. and the difference less than 9 cm.

If you take any three points, and rule straight lines with your straight-edge to join them, the figure you make is called a triangle, and the three lines you have ruled are its sides. Make a triangle having sides 8 cm., 10 cm., and 13 cm. long, another with sides 8, 10, and 17 cm. long, and, if you can, four more with sides (1) 8, 10, 8 cm. ; (2) 8, 10, 19 cm. ; (3) 8, 10, 7 cm. ; (4) 8, 10, 1 cm. Try making other triangles with different sides till you can give a rule to distinguish the cases in which the triangle can be made ; and justify your rule. The rule is important enough to be called a proposition, and may be stated thus:—Any two sides of a triangle are together greater than the third.

5. *Points of the Compass.* The terms north and east have been used. How do you know which direction is north and which east? Point to the south. How do you know?—It is the direction in which you must walk to go towards the sun at midday.

Where does the sun rise and where set?—Roughly, it rises in the east and sets in the west.

If you face south and then turn in the opposite direction, which way are you facing?—North.

Does any one know the Pole Star and how it is used to fix directions? (The Pole Star is not necessary, and need not be followed up if no pupil knows it.)

If you face south and then turn halfway round to the north, which way are you facing?—East or west. In turning halfway round to the north you have turned through an angle called a right angle. Point out any right angles you see about the room, the tables, desks, &c. Through how many right angles do you turn in turning from south to north? And in turning from south back to south again? What fraction is a right angle of the whole angle you turn through in turning from south back to south again? What fraction is a right angle of the angle you turn through in turning round from any position till you again face in the same direction?

Fold a sheet of paper, and call two points on the crease *A* and *B*. Open the sheet out and fold it again so that *A* and *B* fall together. Open it out again. How many angles do the creases make?—Four. And are they all equal?—Yes, because they fit together. Then what angles do they make with one another?—Right angles.

Take two creases that form a right angle and fold them on one another. What angle does the new crease make with the former two?—Half a right angle. By folding make an angle a quarter of a right angle.

Note for the Teacher. If any pupil raises the question whether the size of an angle depends on the length of the arms, the question must be discussed. But the question should be left alone till a pupil raises it. It is a general principle not to point out logical difficulties to a pupil. It is a waste of time to explain a difficulty that the pupil does not feel.

6. Halves and quarters of a right angle have been mentioned. In shopping we avoid fractions of a shilling by using a smaller unit, the penny. In measuring we avoid fractions of a metre by using a smaller unit, the centimetre. So with angles it is convenient to have a smaller unit than the right angle. The smaller unit is called a degree, and

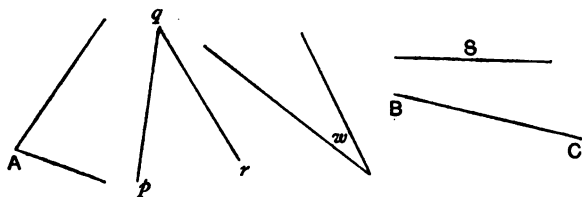


FIG. 1.

90 of them make a right angle. Look at your protractors; they are marked or graduated in degrees. Draw half a dozen angles at random, and measure them in degrees.

How many degrees are there in the first two angles together? Make an angle containing this number of degrees. Make another angle as big as the first three angles that you made.

In turning from north to east, through how many degrees do you turn? And from north to south? And from north to north again? And in turning from north halfway to east?

An angle may be indicated by a single letter or by three letters, as A , w , pqr , in Fig. 1. A straight line may be indicated by a single letter or by two, as S , BC . Care must be taken that the sign used singles out one angle or line.

7. *The Circle.* Draw a circle and fold the paper so that the crease cuts the circle. Open it out again. Call the two points of cutting C and D , and call the centre of the circle A . Draw the radii AC and AD , and fold again so that these radii lie along one another. How do C and D fall? Open out again and call the point where the creases cross O . What are the angles at O ?—Right angles, for they all fit when folded together. What do you know of the lengths CO and OD ?—They are equal because they fit together.

8. The straight line joining two points on a circle is called a chord. Draw any circle and lay in it a chord CD (Fig. 2). Bisect the chord at O by folding C on D . Join O to A , the centre of the circle, and measure the angles AOO and AOD . Repeat with several other chords.

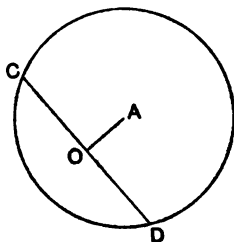


FIG. 2.

Again draw a circle and lay in it a chord CD . Draw a line bisecting the chord at right angles by folding C on D and creasing the paper. How does the centre A lie with respect to this perpendicular bisector? (When two lines meet at right angles they are said to be perpendicular to one another.) Repeat with several other chords.

9. Can you tell from your drawing whether the centre A lies within a millimetre of the perpendicular bisector of the chord? Within 0.1 of a mm.? Within 0.01 mm.? And in the former case, when the centre A was joined to the middle point O of the chord CD , can you tell whether the angle COA differs from a right angle by 1 degree? By 0.1 of a

degree? There is a limit to the accuracy of a conclusion made from drawing. And can you predict whether the same result would be true for additional chords if you drew them?—Not with certainty, though the result seems likely.

If we want an accurate conclusion, and one we know to be true for all cases, we need some other method than this experimental one. Have we in this case another method? Return and compare the discussion in Art. 7. In that there was no measuring, nor did anything depend on the accuracy of our folding or on the particular circle or radii chosen; and we may rely on the general truth of the conclusion. Thus we know that—the straight line joining the centre of a circle to the middle point of a chord is perpendicular to the chord; and the perpendicular bisector of a chord passes through the centre of the circle.

10. *The Isosceles Triangle.* Draw any triangle ACD , having the sides AC and AD equal. A triangle with two sides equal is called an isosceles triangle, and the third side is called the base. Measure the angles at C and D . Repeat with three or four other isosceles triangles. What do you observe about the angles C and D in each case?

Again, draw any straight line CD , and with your protractor draw lines from C and D making the same angle with CD and meeting in A , and so forming a triangle. Measure the lines AC and AD . Repeat with three or four other figures. What experimental result have you reached?

11. If in Art. 7 we began without a circle, but with only two equal lines AC and AD , would there be any difference in our conclusions? What do you know of the triangle ACD made by joining C and D ?—It has two sides equal.

Consider the triangle when AC and AD are folded together. What can you tell about it, and what of the angles at C and D ?—They are equal because they fit together.

State this as a proposition.—The angles at the base of an isosceles triangle are equal.

Now begin with a straight line CD (Fig. 3), and draw lines CE and DF from C and D on the same side of CD and making equal angles with it. Fold C on D . How do CE and DF fall?—Together, since the angles are

equal. If, then, these lines are produced till they meet the crease, what happens?—They meet the crease in the same point. Call it G . Then we have a triangle

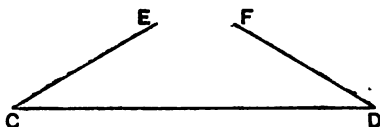


FIG. 3.

GCD in which we made the angles C and D equal. What do we know of the sides?— GC is equal to GD because they fit. This equality may be written

$$GC = GD,$$

the sign $=$ being a short way of writing 'is equal to'. State your result as a proposition.—If a triangle has two of its angles equal, the two sides opposite to these angles are equal.

12. Let us return to the trees A and P , and the two circles with these as centres, at one of whose intersections the cache must lie. Call these intersections C and D (Fig. 4). Draw the figure and fold C on

D , so as to get the crease that bisects CD at right angles. What do we know of this crease in connexion with the circle whose centre is A ? And in connexion with the circle whose centre is P ?—We know that the crease passes through A and through P . So that the perpendicular bisector of a chord that is common to two circles passes through the centres of the circles.

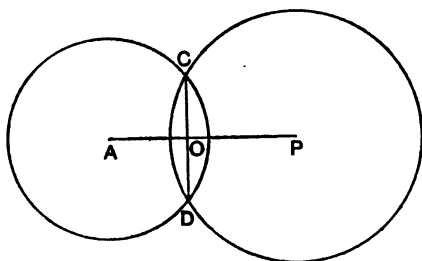


FIG. 4.

13. If we had found the middle point O of CD and joined

it to A and P ?—Then AO is perpendicular to CD (Art. 9), and so lies along the crease made by folding C on D . So does PO , so that AO and OP lie along the same crease and make a straight line, the straight line that joins A and P . So that the straight line joining the centres of two circles bisects the common chord of the two circles at right angles.*

When you see anything in a mirror the reflection or picture or image seems to be behind the mirror right opposite to the original and at the same distance from the mirror. Now in Fig. 4 C and D are on opposite sides of the line AP , right opposite to one another, and at the same distance from it; so we call C the image of D , or D the image of C . And generally, if two points lie so that the line joining them is bisected perpendicularly by another line, they are called the images of one another in that line.

Now draw one circle on ordinary paper and another on tracing paper. Put the tracing paper on the top of the other and move it about so that one circle is sometimes inside the other, sometimes outside, and sometimes they overlap. How does the line of centres lie with respect to the common chord in every position? And in a position when the circles are just ceasing to overlap, how does the common chord lie?—It has become shortened to nothing, and the two points of intersection of the circles have become one.

And how does this single point lie with respect to the centres of the circles?—It is in line with them, since the middle point of the chord is always in line with them.

So we have found that when two circles meet in only one point (or, in other words, when they touch one another) this one point (the point of contact) is in line with the two centres.

14. Suppose, now, that Fig. 2 (p. 5) is drawn in copying ink and folded along the chord CD so that every point of the figure prints itself off on the other side of CD ; or what is the same thing, suppose the image in CD of every point of the figure drawn. What kind of figure have we then?—Two

* On the Euclidian plan reference would be made here to the axiom that two straight lines cannot enclose a space; but such refinements are out of place with beginners.

equal circles with centres A and a , cutting one another in C and D , O the middle point of CD being joined to A and a (Fig. 5).

And, as before, what do we know by considering the radii AC and AD folded together? And by considering the radii aC and aD folded together?—We know the four angles at O are right angles, and AOa is a straight line.

How then can we draw a perpendicular from A on CD ?—Find a , the image of A , by folding, and join Aa .

And what do we know of the perpendicular from the centre of the circle on a chord?—We know that the perpendicular let fall from the centre of a circle on a chord bisects the chord.

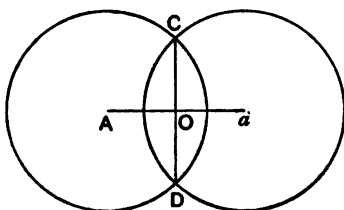


FIG. 5.

15. *Other methods of fixing the cache.* Can you suggest any other measurements that might be made to fix the position of the cache, instead of those already used? The pupils may suggest the following; whether they do or not is immaterial, as these methods may more naturally be discussed later—

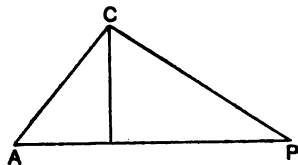


FIG. 6.

1. Distance AC and angle PAC (Fig. 6).
2. Angles PAC and APC .
3. Distance AC and compass bearing of AC .
4. Go a suitable distance along AP , and then a suitable distance out at right angles to AP .

16. As much can be learnt from the discussion of unsuitable suggestions as of suitable, and the following may be discussed if any pupil suggests it:—Would it do to measure the distances of the cache from the foot A of the tree and from a point B some distance up the trunk, say 5 feet?

Assume a position for the cache C , and measure its

distances from A and B , in the garden if possible. If the garden is impossible, take B on the wall of the room, or on a desk, and A right below it on the floor. What do you know of the position of C from the measured distance from A (say 7 feet)? Mark all the points 7 feet from A . These points altogether make what is called the locus of points 7 feet from A . Then take the measured distance from B and mark all points on the floor at this distance from B , another locus. How do these two loci lie?

Assume another position for the cache and proceed as before. Assume additional positions for the cache till the pupils discover and experimentally satisfy themselves that in each case the two loci are the same, so that the second of them gives no further information (Fig. 7).

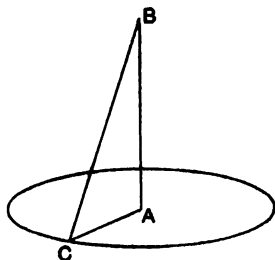


FIG. 7.

Taking B as before 5 feet above A , assume C in succession 1, 2, 3, ... 10 feet from A , measure the distance CB in each case, and make a table showing (1) the length of AC in feet for each position, (2) the length of BC corresponding, and (3) the number of feet in AC divided by the number of feet in BC given as a decimal to two places.

17. *Note.* The work should throughout be done by the pupil, not by the teacher. In drawing and measuring the pupil cannot avoid doing the work; in the matter of conclusions from the drawing and measuring the teacher is tempted to tell the conclusion instead of leading the pupils, by suitable questions, to discover the conclusion. The general reasoning should also be that of the pupils with a minimum of leading by the teacher. A necessary preliminary to the drawing of conclusions and to general reasoning is a certain amount of concrete mathematical experience. If the pupils are slow at reasoning it will be well to increase the amount of drawing and measuring, and to continue the course chiefly experimentally, and then to return to reason out the results with the help of the concrete experience thus

gained. To repeat the words of another's reasoning is not to reason.

18. *Example.* A recluse lives in hiding, and communicates with his friends through two post offices, A and B , of which B lies 14 miles north of A . The hiding-place H is supposed to be at the same distance from these two post offices. Make a map on a scale of 1 cm. to 1 mile, showing A and B ; and supposing H to be 8 miles from each of these points, mark where it might be (Fig. 8). Do the same, supposing H 9 miles from each point, then 10 miles and so on. By taking a great number of points we get a line, and we know that if H is to be equidistant from A and B it must lie on this line. What sort of line is it and how does it lie? If the pupils do not yet know, take different positions for A and B , and again find the line on which H would lie, until the position of the line is made obvious.

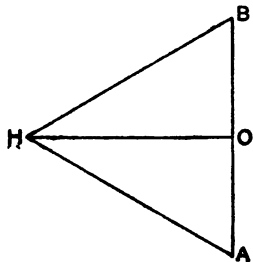


FIG. 8.

Now consider H to be at any one of these positions and join it to O , the middle point of AB . What do you know of HO ? Can you prove it by folding?—Fold HA and HB together, and the crease made lies along HO while A and B fall together; as we saw before. So that HO lies along the crease made by folding A on B , that is, HO is the perpendicular bisector of AB . And this is true no matter which position we take for H ; so that H always lies on this perpendicular bisector.

Can you state the result in mathematical form; what are the conditions and what the conclusion?—The locus of points equidistant from two fixed points is the perpendicular bisector of the line joining the points.

It is also known that the recluse returns home from A by travelling north-north-west. Now the direction midway between north and west is called north-west (Fig. 9), and the direction midway between north and north-west is called north-north-west. Find these two directions on your map

by folding. On what line do we know that the hiding-place *H* lies from the fact that he travels NNW. to get home?

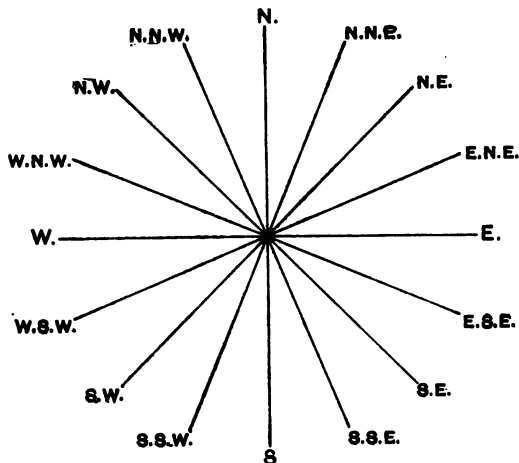


FIG. 9.

Lastly, from the two conditions, mark on your map the spot at which the recluse's friends should look for him.

Constructions.

19. In the preceding article it was found that the locus of points equidistant from two given points *A* and *B* is the perpendicular bisector of the line *AB*. How could you draw this perpendicular bisector?—By folding *A* on *B*.

Could you do it without folding?—Yes, by finding enough points on the perpendicular bisector and joining them.

How would you find a point on it?—It has to be at the same distance from *A* and *B*, so we open our compasses to any convenient radius and draw two circles with centres at *A* and *B*. The point *H*, where they cut one another, is a point on the perpendicular bisector.

How many points do you need?—Two, because two are enough to settle the position of a line

drawn through them. How will you find a second point K ?—In the same way as H ; or simply take the second intersection of the first pair of circles. Then join HK and we have the perpendicular bisector of AB (Fig. 10).

If you had a line AB and wanted to bisect it, that is, to find its middle point, what would you do?—Draw the perpendicular bisector; the point O where it cuts AB is the middle point. Or we could measure AB ; suppose it 8.81 cm.; from this find half the length, namely 4.15 cm., and measure off this distance from one end of the line.

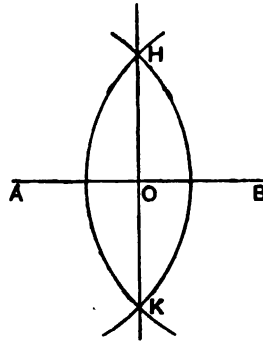


FIG. 10.

20. Could you draw a perpendicular to a line CD from a point E that does not lie on the line?—We might fold the paper so that the crease passes through E , while one part of the line CD falls along another part. Then all the four angles where the crease crosses CD will fit if folded on one another, so that they are right angles and the crease is the perpendicular we want.

Could you draw the perpendicular without folding? Could you find on CD two points, P and Q , such that E would

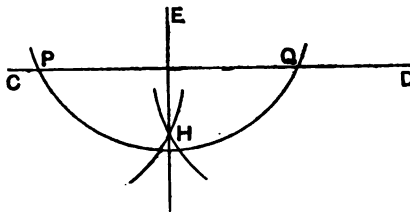


FIG. 11.

lie on the perpendicular bisector of PQ ?—With E as centre, draw any circle that cuts CD in two points, and call

these P and Q . Then E is equidistant from P and Q so that the perpendicular bisector of PQ passes through E (Fig. 11). How many points on this bisector do you need in order to draw it?—One in addition to E . Draw two equal circles, with centres at P and Q , and call one of the points where they cut one another H . Then EH is the perpendicular bisector of PQ , that is, it is the perpendicular from E on CD .

How would you draw at E a perpendicular to CD when

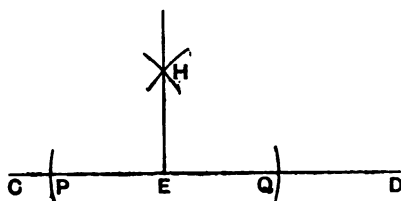


FIG. 12.

E lies on CD ? Will the same method do?—Yes, find P and Q equidistant from E , then find H not on CD and equidistant from P and Q ; and EH is the perpendicular (Fig. 12).

How would you use your protractor to draw a perpendicular to CD at a point E in it? Do so. Some protractors can be used to draw the perpendicular when E is not on CD . Can yours be so used?

21. Converse Propositions. We have found properties of circles so related that a condition and the conclusion of one become respectively the conclusion and a condition of the other. Thus we had the trio:—(1) The perpendicular bisector of a chord of a circle goes through the centre. (2) The line joining the middle point of the chord to the centre is at right angles to the chord. (3) The perpendicular from the centre on the chord bisects it.

Any one of these is said to be the converse of any other. Another pair of converse propositions we had was:—(1) The line joining the centres of two circles bisects their common chord at right angles. (2) The perpendicular bisector of

the common chord of two circles passes through the centres of the circles.

The following are statements related in the same way. Discuss which of them are true and which false. Draw a circle of radius 5 cm. and centre A and take a point B somewhere inside the circle (Fig. 13). Is it true that every point more than 5 cm. from A lies outside the circle; and that every point outside this circle is more than 5 cm. from A ? Is it true that every point more than 5 cm. from B lies outside the circle; and that every point outside the circle is more than 5 cm. from B ? Is it true that every point more than 10 cm. from A is outside the circle; and that every point outside the circle is more than 10 cm. from A ? Is it true that every point more than 10 cm. from B is outside the circle; and that every point outside the circle is more than 10 cm. from B ?

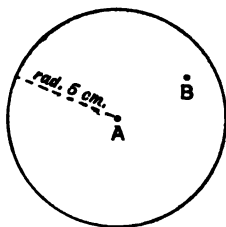


FIG. 13.

Does it appear whether, from the truth or falsity of a statement, you can tell whether a converse of the statement is true or false?

EXERCISES.

1. The positions of four points are specified thus: from O to A is 1.6 miles north and 4.2 east; from A to B is 3.1 north and 2.3 west; from B to C is 2 north and 3.3 west; and from C to D is 3.7 south and 1.4 east. Show the points $OABCD$ on the scale of 1 inch to one mile.

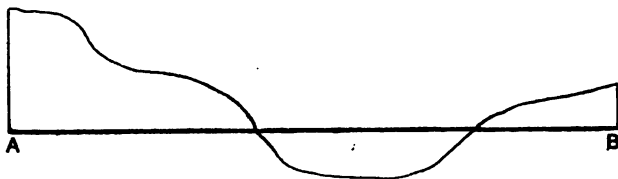


FIG. 14.

2. In Fig. 14 the thick line AB represents a level railway which is carried partly through cuttings and partly on an embank-

ment. The irregular line represents the section of the ground. If the scale of horizontal distances is 1 inch to a mile, and the scale of vertical distances is 1 inch to 100 feet, draw up a table showing the height of the ground above the level of the railway at intervals of half a mile starting from A , and indicating points where the ground is below the level of the railway by prefixing the sign $-$.

3. Prick off Fig. 15; in which A and B are two forts. A body of soldiers at Y are just out of range of each fort and want to get as near X as possible without going any nearer to either fort. Show the point they would go to, and draw the shortest path by which they could go. Draw the straight part of the path by laying your straight-edge against the circle and through their destination.

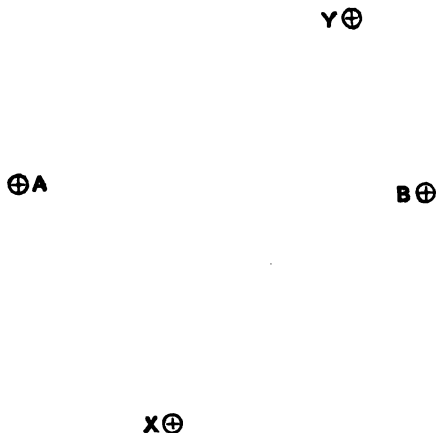


FIG. 15.

4. AB is a tube which revolves uniformly on a table about its middle point O , that is, turning through the same angle every second. If a marble moves from one end of the tube to the other at a uniform speed (that is, travelling the same distance every second), whilst the tube makes a complete revolution, draw the path of the marble. Represent the tube by a line 5 inches long.

5. From a corner of a sheet of paper mark off 20 cm. along one edge to A and 13.4 cm. along the other to B . Fold over the corner along the line AB , and mark on the page the point O where the corner falls. Open out the fold again and join O to the corner by a straight line. Prove shortly that this straight line is perpendicular to AB .

CHAPTER II

POSITION OF CHAIR ON SCHOOLROOM FLOOR

1. A CHAIR stands in a certain position on the schoolroom floor. What measurements would you take so that if the chair is moved it may be placed again in its former position?

Discussion will soon show that this resolves itself into two questions, how to fix one leg of the chair, and how many legs must be fixed. The former question is the problem of the cache over again, and the same solution may be offered. The teacher accepts this and asks for another. A likely one is to measure the distances of the leg from the walls.

Proceed to measure the distance from a wall, say the north wall. From what point of the wall will you measure? The suggestion may be made of the nearest point of the wall. Measure from various points and measure the angles these various distances make with the wall. Make a table of the various distances and the corresponding angles. To measure at right angles is thus suggested; or it may have been suggested at first.

Suppose the leg of the chair known to be 4·6 feet from the north wall, measured at right angles. What locus does this give? on what line does the leg lie? Find by drawing; draw a number of lines 4·6 feet long perpendicular to the wall, either on the floor or to a scale of 1 cm. to 1 foot on paper. What is the locus of the ends of these perpendiculars?—It looks like a straight line. Test it with your straight-edge.

2. *Parallels.* Fold a sheet of paper and call the crease you make *A*. Fold the crease *A* on itself in two different places, making creases *B* and *C* (Fig. 1). What angles do these creases *B* and *C* make with *A*?—Right angles, since the

four angles at either intersection all fit together.

Now fold the crease *B* on itself in several places, making new creases *D, E, F, ...* across *B* and *C*.

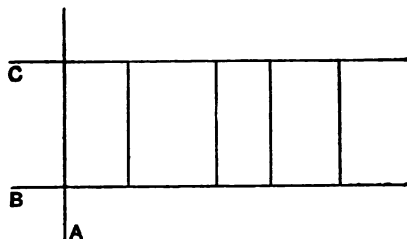


FIG. 1.

Measure and tabulate the lengths of these last creases that lie between *B* and *C*. What angles do these creases *D, E, ...* make with *B*? How do you know? Measure and tabulate the angles that *D, E, ...* make with *C*. What experimental result do your measurements give?—That the creases *D, E, ...* cut *B* and *C* pretty exactly at right angles, and that the distances between *B* and *C* measured along them are pretty exactly the same.

The creases *B* and *C* are said to be parallel. To be made mathematically parallel in this way they would need to have the angles between them and the creases *D, E, ...* made exactly right and the distances apart along all the creases *D, E, ...* made exactly equal.

Repeat this experiment by drawing, that is, draw any line *A*, draw two others *B* and *C* at right angles to it, and at various points on *B* draw perpendiculars to meet *C*. Measure the lines and angles as before.

Again draw any straight line, and at a number of points on it draw perpendiculars, all of the same length and all on the same side of the line. Join their ends. What is the result?—The ends all lie on a straight line.

Mention any cases from everyday life of parallel lines.—The edges of boards of the floor; the two side pieces of a ladder; the two rails of a straight railway track; the courses of brick in a wall.

NOTE. The existence of parallel lines is an experimental fact. All systems of demonstrative geometry assume it in some form, though often not explicitly.

8. To return to our problem: it is now known that the chair-leg lies on a parallel to the north wall 4·6 feet from it.

How would you draw this parallel?—We have had one method, by finding a number of points on it; a simpler way is to measure 4·6 feet along the neighbouring walls and to join the points thus found.

Is this enough to fix the leg?—We need the distance from a second wall.

Is the leg then fixed? Will any second wall do?—It must be the east or the west wall, and the leg is then fixed, for this gives a second locus meeting the former locus in one point. The south wall would not do, for the locus it would give is the locus we have already. So that the position of the leg is fixed by its distances from two walls that meet.

4. Having fixed one leg, say the right-front-leg, how will you fix the chair?—By fixing another leg, say the left-front-leg.

How?—In the same way. Is the chair now fixed?—Yes, for you can place it in position from these data.

How many measurements have you made to fix the chair? Could you have done with fewer? When you have fixed one leg do you know anything about the position of the second leg?—If the first leg is set down on the proper spot *P* (Fig. 2), the second lies somewhere on a circle with this spot *P* as centre and with the distance between the legs as radius; so that we need only one more measurement. If we know the distance of the second leg from the north wall, its position must be *Q* or *R*, one of the points where the two loci meet. And if we know also that the second leg is nearer than the first to the east wall, its position is fixed without ambiguity.

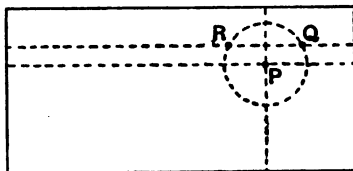


FIG. 2.

What information then do we need to fix the chair?—Three measurements, in addition to a means of distinguishing between *Q* and *R*.

Ex.1. You are given that the right-front-leg is 4·6 feet from the north wall and 5·1 feet from the east, the other front leg is 4·8 feet from the north wall, and the distance between the legs is 1·5 feet.

Suppose 1 cm. represents 1 foot and a page of your book represents the floor; and mark the position of the chair.

If you had been given that the left-front-leg was 6.8 feet from the north wall, could you have marked its position? Why not? What condition must hold among the measurements?

5. When you have the chair in position can you move it at all without shifting the two legs from their positions on the floor?—We may tilt the chair. Then state more exactly what you have found.—Provided the chair stands on the floor the three measurements and the means of distinguishing between *Q* and *R* settle the position of the chair.

What measurements would be necessary if the chair is not to be compelled to stand on the floor? Think this over, and we will return later to the discussion of such a problem.

6. Is any other measurement possible for the second leg, instead of the distance from a wall, when you have the first leg fixed? The angle the line joining the two legs makes

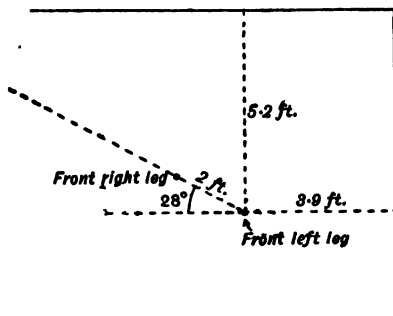


FIG. 3.

with a wall or with a parallel to a wall would do. And various other suggestions may be made.

From the sketch given (Fig. 8) draw the chair in position on the same scale as before.

7. A boy once made a cache in the corner of a rectangular lawn, and to keep a note of the position he made the

drawing in Fig. 4. But, as boys and even men sometimes will, he forgot to make a note of the scale of his drawing, whether an inch represented a foot, or every distance was $\frac{1}{10}$ of the true distance, or whatever else it might be. What does his drawing tell him of the position of the cache? Measure in the drawing the distance of the cache from the two edges of the lawn.

He knows only that the cache is at the same distance from the two edges. Draw the lawn to some definite scale, this time marking the scale, and draw the locus of all points where the cache might lie. How do you mark the point that is 8 cm. (say) from each edge?—By measuring 8 cm. from O

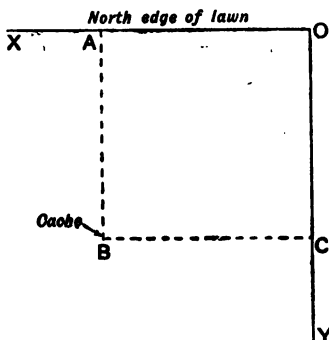


FIG. 4.

along the north edge to A , and there drawing a perpendicular AB 8 cm. long. Or by measuring along the east edge to C and drawing CB perpendicular and 8 cm. long. Or by measuring 8 cm. along to A and to C , and drawing perpendiculars long enough to meet.

When the last method is used, do we know that the point B where the perpendiculars meet is 8 cm. from each edge?—Yes, from our experimental result we know that if OA and CB are both perpendicular to OC , their distance apart is the same all along as between O and C , that is, it is 8 cm.

If now the two edges OX and OY are folded together, how do A and C fall? And how do the perpendiculars at A and C fall? Then how does the point B where they meet lie?—On the crease.

8. Does this result depend on the particular length 8 cm. that we used?—No, whatever the length B would lie on the crease.

What then is the locus of possible positions of the cache?—It is the crease.

Compare the angles the crease makes with OX and OY by measuring.

They are equal as far as measurement can tell.

Can you give any other reason?—They are equal because they

fit when the paper is folded. So that the locus of possible positions for the cache is the bisector of the angle between the edges of the lawn.

State the geometrical property without reference to lawns or caches.—If two straight lines OX and OY are at right angles the locus of all points equidistant from them is the bisector of the angle XOY .

9. Now produce XO , YO , and BO so that we have XOZ and YOW crossing at right angles, while BOQ bisects the angle XOY (Fig. 5). What can we say of this figure? Are points on OQ equidistant from OW and OZ , and does OQ bisect the angle ZOW ? And do all points equidistant from XZ and YW lie on BQ ?

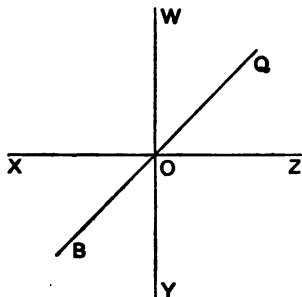


FIG. 5.

Measure all the angles at O with your protractor and write their values on the figure. When you stand facing north and turn about to face south, through how many degrees do you turn? And if from any position you turn about to face the opposite way? Does it matter whether you turn to the right or to the left? Suppose that in Fig. 5 you stand

at O facing along OX and then turn about to face the opposite way; write down the angles that make up the 180 degrees you have turned through.—They are XOW , WOQ , and QOZ , if we turn to the right. Do this for every line of the figure.

You found the sizes of all these angles by measuring. But the value of your results depends on the accuracy of your drawing and of your measuring, which can never be very great. Can you by reasoning tell what results you would get if you could draw and measure the figure with perfect accuracy? What would the angle XOY be on a perfect figure? And how did you get the line OB ?—By folding OX on OY .

What, then, do you know of the angles XOB and YOB on a perfect figure?—They are equal, and

together they make a right angle or 90 degrees ; so that each is 45 degrees.

Can you tell the size of the angle XOW ?—It is 90 degrees, for the turn from OY to OX is 90 degrees and the turn from OY to OW is 180 degrees.

And what do you know of the sum of the angles BOX , XOW , and WOQ ?—It is 180 degrees. And of these three angles you know two ; what, then, is the third WOQ ?—It must be 45 degrees to make the sum 180 degrees.

10. To write this down in the ordinary way takes a long time, and some contractions or symbols are used to shorten the writing. Thus—

angle is denoted	by $\angle$
degree	by $^{\circ}$
equals or is equal to	by $=$
together with or plus	by $+$
less or minus	by $-$
perpendicular	by $\perp$
parallel	by $\parallel$
triangle	by $\triangle$
therefore	by $\therefore$

and by the help of these symbols our reasoning may be stated thus—

$$\begin{aligned}\angle XOY &= 90^{\circ} \\ \angle XOY + \angle BOY &= 90^{\circ} \text{ and } \angle XOY = \angle BOY \\ \therefore \angle XOY &= \angle BOY = 45^{\circ} \\ \angle YOX + \angle XOW &= 180^{\circ} \text{ and } \angle YOX = 90^{\circ} \\ \therefore \angle XOW &= 180^{\circ} - 90^{\circ} = 90^{\circ}\end{aligned}$$

and so on. But these contractions should be sparingly used at first as there is danger of concentrating the attention on the contractions instead of on the reasoning. The value of symbols is best realized if the pupils are allowed to do their writing in longhand till they begin to realize how tedious it is ; they may then be encouraged to invent symbols for themselves and to use their own inventions for a time.

11. Now find the size of each angle of the figure by similar reasoning, and compare these results with the measured results. Can you now answer any of the questions at the beginning of article 9? Are all points on OQ equidistant from OZ and OW ?—Yes; OQ bisects the angle ZOW and folding along OQ brings OZ and OW together, so that if any length is measured along OZ and OW , and perpendiculars drawn, they meet on OQ and have the same length; just as we found already for OB , OX and OY . So that we now know that all points on BOQ are equidistant from XOZ and YOW .

The question whether OQ bisects $\angle WOZ$ is already answered.

Do all points equidistant from XOZ and YOW lie on the line BOQ ? Find a number of such points by drawing, and then answer the question.—No, all such points do not lie on BOQ ; some lie on the line that bisects the angles XOW and YOZ .

Here, then, we have two converse propositions. Are they both true, or both untrue, or what?

Can you sum up what we have found in a single proposition about a locus of points? A little discussion should lead to the statement that: The locus of points equidistant from two lines intersecting at right angles is the two lines that bisect the four angles made by the intersecting lines.

Ex. 2. Use the result of Article 8 to bisect a right angle by the help of ruler and compasses.

12. Suppose now that the lawn in Art. 7 is not rectangular, and that as before you know the cache is equidistant from two edges, the distances being measured perpendicular to the edges. Go over the work of Arts. 7–11, altering it to suit the altered condition.

13. Draw two straight lines AOB and COD crossing at O , and measure all four angles. The angles AOC and BOD are called vertically opposite angles. The angles AOD and BOC also are called vertically opposite. You have met with several such pairs of angles. Can you state and prove any proposition about a pair of vertically opposite angles?—The vertically opposite angles made by two intersecting lines are equal.

EXERCISES.

3. In article 12 it is found that the bisector OB of any angle XOY may be found by laying off equal lengths OA and OC along OX and OY and drawing perpendiculars at A and C to meet in B . Find out if it would do to join O to the middle point D of AC ; would this line OD be the bisector of the angle?

4. Make a map of the schoolroom and the desks, &c., in it. Use a scale of 1 cm. to 1 ft., 1 in. to 1 ft., 1 cm. to 1 metre, or any other that is convenient.

5. How would you fix the position of a pencil that lies on your desk? A book? A triangle cut out of paper? An irregular sheet of paper?

6. How would you draw a line parallel to a line that is given you? How would you do it if the parallel was to go through a given point?

7. Two straight railroads, AOO and BOD , cross at O at right angles. A is $1\frac{1}{2}$ miles, and B is $1\frac{1}{2}$ miles distant from O . Two trains pass A and B respectively, at noon, moving towards O at the uniform rates of 20 and 30 miles an hour. Find by measurement, and tabulate, the distances apart of the trains at noon, and at 1, 2, 3, . . . 7 and 8 minutes after noon. By examination of your table of distances, state as accurately as you can the time when the trains were nearest to one another. Take O about the middle of the page, and work to a scale of 2 inches to a mile.

8. Draw on tracing-paper a straight line SH six inches long: mark on it a point B two inches distant from S . On a page of your book draw two straight lines XOP and OQ intersecting at right angles, XOP close to one side of the page and OQ across the middle of the page. Bring the tracing-paper over these lines, moving it about so that S passes up and down XOP and at the same time H moves to and fro along OQ . Prick on the paper the points over which B passes and then draw a freehand smooth curve through these points.

9. A square $ABCD$, measuring 3 cm. in the side, rolls without sliding along a line XY , turning about B , C and D in succession. Draw (full size) the path of the point A from the position in which it is leaving XY till it again lies on XY . (A square has all its sides equal and all its angles right angles.)

10. Draw two circles of 1 inch radius with their centres $2\frac{1}{2}$ inches apart. On tracing-paper draw two straight lines each $1\frac{1}{2}$ inches long bisecting each other at right angles. By pricking through the tracing-paper and drawing through the points a smooth curve, determine the paths of the ends of one of the straight lines while

the ends of the other move along the circumferences of the two circles one end on each.

11. A point P is known to be at least 4.6 cm. from a point A of a straight line; shade the area in which P may not lie. P is also to be at least 4.6 cm. from another point B of the line; shade the additional prohibited area. Take more points $C, D, \dots$ on the line, and suppose P to be at least 4.6 cm. from each, and shade the prohibited area. Finally, find experimentally and approximately the form of the prohibited area if P is to be at least 4.6 cm. from every point of the line.

CHAPTER III

AREAS

1. THE determination of areas is a problem of great antiquity. Many races have passed through a stage of development in which the land of a community was periodically divided into a number of equal parcels; and the name Geometry itself means land-surveying.

Do you know any units in terms of which areas are measured?—Acre, hectare, square foot, square decimetre, square inch, square centimetre, and others.

Draw as many of these units as your paper is big enough to hold, and cut them out to use for measuring. Take a sheet of paper of any form, and see how many square inches you can mark out on it. Does this tell the area?—Yes, except for the irregular pieces left at the edges.

Are there no pieces left over except at the edges?—We can mark out the inch-squares so that there will be no pieces left over except at the edges.

Could you measure the playground in the same way with a square foot as unit?—Yes, but it would be somewhat laborious.

Let us see if we cannot find easier ways.

2. In measuring the sheet of paper, do you need to lay the unit-area on the sheet every time to mark out the units there?—We could simply rule the sheet into squares, or we could place it on squared paper and trace its boundary on to the squared paper.

Is there any choice as to where you mark out the first square?—We could use any straight side of the sheet as the side of squares, and if the sheet was rectangular * one square should be placed in a corner of the sheet.

* The sheet is rectangular when its four angles are all right angles.

Take a rectangular sheet and rule it out as far as you can into inch squares. Cut off the strips left over, and state the relation between the area, the length, and the breadth of the rectangle you now have.—The area in square inches is the length in inches multiplied by the breadth in inches.

Does this result depend on the particular length and breadth, or would it be true for others? And if the length is l inches and the breadth b inches, what is the area?—It is $l \times b$ square inches, provided l and b are whole numbers.

3. Let us now replace the strips that were cut away from the rectangle and try to find the area of the original sheet of paper. No inch-squares can be marked out on these strips, and we must use a smaller unit. Let us take as smaller unit a strip such that ten of them placed side by side form an inch square. Mark out this strip-unit as often as you can on the strips that were cut away. If any area still remains that has not been included take a still smaller unit, say a little square, ten of which will make a strip-unit.

Thus, for instance, if the sheet of paper measures 12·7 inches by 8·2 inches, the area consists of

$$\begin{aligned} &12 \times 8 \text{ inch squares,} \\ &12 \times 2 + 8 \times 7 \text{ strip-units,} \\ &7 \times 2 \text{ of the smaller squares.} \end{aligned}$$

How many strip-units are there?—80. If you were given the expression

$$12 \times 2 + 8 \times 7$$

without knowing how it was arrived at, would its meaning be clear?—No, unless there is an agreement about the order in which the addition and multiplication are to be performed; if we began by adding the 2 and the 8, the expression would mean 840. The expression then may mean the sum of the two products 12×2 and 8×7 , or it may mean the product of the three quantities 12, 2 + 8, and 7. Brackets may be used to avoid ambiguity; thus

$$(12 \times 2) + (8 \times 7)$$

means the sum of the two products 12×2 and 8×7 , while

$$12 \times (2 + 8) \times 7$$

means the product of the three numbers 12, $2 + 8$, and 7. There can be no doubt about the meaning of either of these, but we can save ourselves some labour in writing if we agree to confine the brackets to one of the two cases. Let us agree that in the absence of brackets or other indication multiplication and division are to precede addition and subtraction. Then when we meet $12 \times 2 + 8 \times 7$ we know that we must first multiply 12 by 2 and 8 by 7 and then add the results. So that with this agreement the expression as it originally stood means 80 without any ambiguity, and if we want to make it mean 840 we must use a bracket.

4. Now express in square inches and fractions the area of the sheet of paper.—It is

$$12 \times 8 + (12 \times 2 + 8 \times 7) \times \frac{1}{16} + 7 \times 2 \times \frac{1}{16}$$

square inches. (Note the use of the bracket; what would this expression with the bracket left out mean?) This expression, which is now recognizable as $12 \cdot 7 \times 8 \cdot 2$, has the value 104·14. For most purposes it is sufficient to give the value as 104, and if the length and breadth of the sheet are only approximate, being measured to the nearest tenth of an inch, the form 104·14 has a misleading appearance of accuracy, and even 104 is not reliable in the unit figure.

If you adopted as unit from the beginning the small square 0·1 of an inch in the side, how many of these units would you have in the rectangle? And what is their equivalent in square inches?

Ex. 1. Make a loop of string, lay it on squared paper, and find the area it encloses by counting squares. Move the loop about and pin it out, to make it enclose as great an area as possible. What does the shape of the loop then look like?

5. Now generalize the formula already obtained for the area of a rectangle in terms of the length and breadth.—The length being l inches and the breadth b inches the area is $l \times b$ square inches, where l and b may contain fractions.

Are these fractions limited to be tenths? Or to be

decimals? And what is the formula if the dimensions are given in centimetres?

6. What is the area of a rectangle measuring a inches and b tenths by c inches and d tenths?—It consists of

$$\begin{array}{ll} a \times c & \text{inch squares,} \\ a \times d + b \times c & \text{strip-units,} \\ b \times d & \text{smaller squares,} \end{array}$$

that is,

$$a \times c + a \times d \times \frac{1}{10} + b \times c \times \frac{1}{10} + b \times d \times \frac{1}{100}$$

square inches.

This form is clumsy, and a shorter way of writing it is desirable. One a little better is

$$a \times c + \frac{a \times d}{10} + \frac{b \times c}{10} + \frac{b \times d}{100},$$

but even this may be improved on. Would it do to drop the sign of multiplication and write ac to mean a multiplied by c ?—It would not do for numerals; we cannot drop the sign in 12×8 , for it would then be 128, that is, one hundred and twenty-eight.

May we drop it except between numerals?—Yes. A system is a little illogical that adopts different conventions for letters and numerals, and this difference of convention may cause difficulty. However, these conventions are universal and very useful, and pupils that realize the difference, and realize that both conventions are no more than conventions, will find no difficulty. We agree, then, that the sign $\times$ may be dropped when no ambiguity results from dropping it, and we write the number of square inches in the rectangle as

$$ac + ad\frac{1}{10} + bc\frac{1}{10} + bd\frac{1}{100},$$

or as

$$ac + \frac{ad}{10} + \frac{bc}{10} + \frac{bd}{100}.$$

Area of a right-angled triangle.

7. Suppose that the playground that we want to measure is shown in Fig. 1, on a scale of 1 to 1000. We saw one way of measuring its area, by marking out a foot square as

often as possible. Can we now shorten this work?—We could mark out a few big rectangles on the playground, measure their length and breadth, and so find their area.

But some pieces will remain over, and what shall we do with these? What is their shape?—They are triangles, two of them right angled (i.e. having one angle a right angle). So that to find an expression for the area of a triangle would be useful.

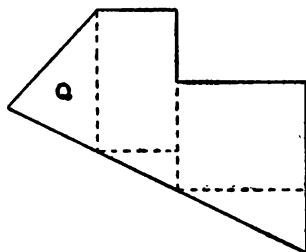


FIG. 1.

Take a rectangular sheet of paper and cut it along a diagonal, that is, along the line joining two opposite corners. What kind of figures have you now?—Two right-angled triangles.

Which of them is greater?—Fitting together shows them equal experimentally, that is, equal as far as observation will tell us.

We will discuss later whether two triangles obtained by a cut accurately along a diagonal of an accurate rectangle will fit accurately, that is, whether the triangles are mathematically equal. What do you know of the area of each triangle?—It is half that of the rectangle.

Now draw any right-angled triangle. Is there any rectangle of which this triangle is half? Draw this rectangle. Express the area of the rectangle in terms of the sides of the triangle.—If the two sides of the triangle that meet at the right angle are p cm. and q cm. long, the area of the rectangle is $p \times q$ sq. cm.

And the area of the triangle itself? —It is $p \times q \div 2$ sq. cm.

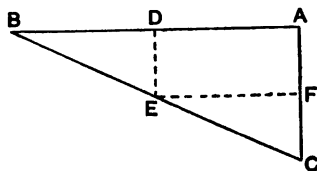


FIG. 2.

8. Take again one of the triangles into which you cut the sheet of paper. Call the corner at the right angle A and the others B and C (Fig. 2). Fold B on to A and fold C on to A , forming creases DE and FE . Measure the sides and angles of the figure $ADEF$. What is the figure $ADEF$?—

As far as observation tells it is a rectangle. And mathematically what do you know of the figure $ADEF$?—That the folding makes the angles at D and F right angles, and makes AD half of AB and AF half of AC , while A was supposed a right angle from the beginning. Later we will discuss whether this is enough to show $ADEF$ a rectangle. Meantime, the sides AB and AC being p and q cm. long, what is the area of the rectangle $ADEF$?—It is

$$\frac{p}{2} \times \frac{q}{2} \text{ sq. cm.} \quad \text{or} \quad \frac{1}{4} pq \text{ sq. cm.}$$

When the corner B of the triangle ABC is folded in on A , is the piece of paper still a triangle?—No. Why?—Because the three boundaries are no longer straight, and a triangle is bounded by three straight lines.

When B and C are folded in on A , how does the piece of paper lie?—By observation it is made up of the rectangle $ADEF$ and the folded-in parts which just cover the rectangle.

Whether this is so mathematically we shall see later. Then what is the area of the triangle?—It is twice the rectangle, or $\frac{1}{2}pq$ sq. cm. Thus our previous result is confirmed.

9. How much of the playground in Fig. 1 are we now in a position to measure?—All but the triangle Q . Can you suggest a method of dealing with the triangle Q ? Can you divide it into right-angled triangles?—Yes, by a perpendicular from a corner on the opposite side.

You are now in a position to find the area of the playground of Fig. 1. Make the necessary measurements of the figure, and, remembering that every length on the figure is 0.001 of the true length, find its area in square feet or in square metres.

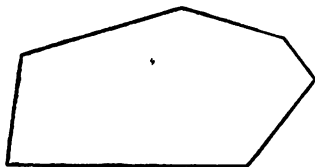


FIG. 3.

If your own playground is simple enough to treat similarly measure it up and find its area.

10. Could you now find the area of a playground such as Fig. 8 shows? Could you divide it into triangles?

And can you find the area of any triangle, no matter what its shape is?

Again, could you find the area of a playground whose boundaries are not straight? You could replace the curved boundaries by one or more straight ones, placing the straight boundaries so that, as far as the eye can judge, as much playground is now left outside as other ground is brought in. Or, if you may not go beyond the boundaries of the playground, you could make a straight-sided figure by joining up a number of points in the boundary, and allow by eye or by further measurements for parts left over.

Ex. 2. Find in acres or hectares the area of a field represented in Fig. 3 on a scale of 1 to 10,000.

Ex. 3. Draw any irregular figure, and find its area. Check your result by using squared paper.

Ex. 4. Find the area of a circle of radius 8 cm. by both methods.

Area of a triangle.

11. Let us return to a triangle of any form. Can you extend the methods used for the right-angled triangle to cover the general case?

We might, as in treating the triangle Q of Fig. 1, divide our triangle into two right-angled triangles. Suppose one side of the triangle b cm. long and the perpendicular from the opposite corner h cm. long, and that the perpendicular divides the base into two parts p and q cm. long. What are the areas of the two right-angled triangles?—They are

$$\frac{h \times p}{2} \text{ sq. cm. and } \frac{h \times q}{2} \text{ sq. cm.} \quad \text{And together?—}$$

$$\frac{h \times b}{2} \text{ sq. cm., since the side } b \text{ is made up of the two parts } p \text{ and } q.$$

How must this be modified if the triangle is obtuse-angled* and the perpendicular h falls not on the side b , but on this side produced?—In this case the triangle dealt with is not the sum but the difference of two right-angled triangles.

* An angle less than 90° is called an acute angle, an angle greater than 90° and less than 180° is called an obtuse angle. A triangle whose angles are all acute is called an acute-angled triangle, and a triangle with one angle obtuse is called an obtuse-angled triangle.

12. Again, take a rectangular sheet of paper ($ABCD$ in Fig. 4) and make two cuts, ED and EC , from a point in

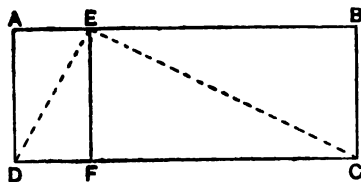


FIG. 4.

one side to the ends of the opposite side. Can you fit the corners cut off on to the remaining triangle CED ? What then is the area of the triangle?—It is half the rectangle, that is, half the length of the rectangle multiplied by the

breadth; the length and breadth and the area being expressed in corresponding units, e.g. inches and square inches or centimetres and square centimetres.

And in terms of the parts of the triangle?—The area is half the base CD multiplied by the height EF , as we found before.

You assume that EF is equal to AD or BC . How do you know?—By the experimental result that parallel lines are at the same distance apart all along.

How would you cut the rectangle to give an obtuse-angled triangle?—Cuts along AF and AC would do.

And how far does the discussion differ for this case?

13. Again, take any paper triangle CDE ; Fig. 4 will serve. Fold the side DC on itself so that the crease passes through E , as EF in Fig. 4. Straighten out, and then fold the corners CDE down on to F . What figure results?—It is a rectangle exactly covered by the folded-in corners.

What are the length and breadth of the rectangle?—They are half the base and height of the triangle. So that as before the area of the triangle is half the product of the base and the height.

Repeat this for the case in which the angle D is obtuse, noticing that till the folding is done an extra piece of paper to carry the point F must be left.

14. Once more draw any triangle, acute- or obtuse-angled, and draw about it a rectangle of which it is half, and so proceed to the expression for the area of a triangle.

In this discussion we have chosen any side of the triangle and called it the base, while we called its distance from the opposite corner the height of the triangle. And as a triangle has three sides, the expression we have found gives three ways of finding the area of a triangle. Draw a triangle, and use the formula in the three ways, and compare your three results. Further, check your results by drawing the same triangle on squared paper and counting squares.

Graphs.

15. The daily newspapers give curves showing the height of the barometer for the preceding day or the preceding week. In these, distances measured from an initial point along a base line running parallel to the lines of print (running horizontally, we may say by an obvious metaphor) represent time elapsed; and distances perpendicular to the base line, that is, parallel to the columns of print, or vertically, represent the height of the barometer. The length of the perpendicular from any point of the base line to the curve gives the height of the barometer at the moment denoted by that point of the base line. A second curve gives the height of the thermometer. To save space it is usual to use the same diagram for the two curves, and to show only the upper part of the diagram, thus omitting the base lines.

Such curves can be more quickly read than a table of numbers giving the same information. They are called graphs. The scales used are of course shown on the figure; without them the curves would lose much of their meaning.

16. Take a sheet of paper 20 cm. in breadth, and place a ruler across it at one end. Slide the ruler along, keeping it all the time parallel to the end of the sheet, till it has moved 20 cm. along the paper. Find and show in a table the area of the paper that the ruler uncovers by moving 2 cm., 4 cm., 6 cm., . . . 18 cm., 20 cm.

Distance ruler has moved, in cm.	2	4	6	8	...
Area uncovered, in sq. cm.	40	80	120	160	...

Now mark a line OX 20 cm. long on squared paper (if you have no paper squared in centimetres, use any convenient scale), and mark along it from O 2 cm., 4 cm., . . . At the point marked 2 cm. raise a perpendicular to OX to represent on a scale of 1 cm. to 40 sq. cm. the area uncovered by the ruler in moving 2 cm., that is, raise a perpendicular of length 1 cm. At the point marked 4 cm. raise a perpendicular to represent on the same scale the area uncovered by the ruler in moving 4 cm., and so on.

If you wanted to show on your diagram the area uncovered when the ruler had moved 1 cm. or 3 cm., how would you proceed?—Erect a perpendicular as before. And for any other distance, for instance, 3.4 cm.?

Through the ends of all these perpendiculars draw a smooth line. This line shows graphically how the uncovered area varies as the ruler moves.

17. Again, suppose the ruler to be returned to its position at the end of the sheet of paper, while a second ruler is placed along the side of the sheet. Let the two rulers now move away, each remaining parallel to its original position, in such a way that the area uncovered is always a square, that is, has its angles right angles and its sides all equal. As before, show in a table and in a graph the variation of the uncovered area as the rulers move. From your graph read off the area of a square whose side is 12.8 cm., and the length of the side of a square of area 178 sq. cm.

18. Two rods 18 cm. and 10 cm. long are hinged together at one pair of ends, and the other two ends are joined by an elastic string. Find by drawing, and show in a graph, how the area of the triangle formed by the two rods and the string varies as the angle between the rods increases, or, in mathematical language, show the area as a function of the angle.

Algebraic expressions.

19. In finding the area of a rectangular piece of ground we treated each dimension as made up of two parts, an integral part and a fractional part. Let us now suppose the

length made up of two parts a cm. long and b cm. long, where a and b may be fractional; and the breadth of two parts c cm. and d cm. long. Sketch the rectangle, and divide it into rectangles which have sides of lengths a b c d cm., as in Fig. 5. Mark on each rectangle its area in sq. cm., and so find an expression for the whole area.—It is in sq. cm.

	a	b
c	$a \times c$	$b \times c$
d	$a \times d$	$b \times d$

FIG. 5.

$$a \times c + a \times d + b \times c + b \times d,$$

or, written in the shorter form,

$$ac + ad + bc + bd.$$

By considering the undivided rectangle you can give another expression for the area. It is the product of $a+b$ and $c+d$, which we agreed to write in the form

$$(a + b) \times (c + d),$$

or

$$(a + b)(c + d).$$

What then do we know of these two expressions?—That they are equal, since each is the number of sq. cm. in a certain area.

20. Is it true that whatever numbers a b c d are

$$(a + b)(c + d) = ac + ad + bc + bd?$$

Is it true only when these numbers are the lengths of lines, or is it true also when you buy $a+b$ loaves at $c+d$ pence each, or, as a particular case, 5 white and 3 brown loaves at $2\frac{1}{2}$ pence each?—We are at liberty to buy the white loaves first and then the brown, and we are at liberty to pay down 2 pence on every loaf and then the remaining halfpenny; which gives us as the price in pence

$$5 \times 2 + 5 \times \frac{1}{2} + 3 \times 2 + 3 \times \frac{1}{2}$$

in addition to the other form, $8 \times 2\frac{1}{2}$ pence. Or we could represent the whole price by a figure, such as Fig. 5, representing a loaf by a centimetre across the page and a

penny by a centimetre down the page ; the partitioned area would then represent the four parts of the price. And the same reasoning applies whatever numbers of loaves are bought and whatever the price per loaf.

May you then conclude that

$$(a + b)(c + d) = ac + ad + bc + bd$$

for all numbers, whatever may be the concrete problem in hand?—It looks as if other cases could be represented in the same way by Fig. 5.

It will be well for you to watch as to the applicability of this formula, when you have occasion to use it. There are kinds of numbers (negative numbers, complex numbers, vector quantities, &c.), some of which you may meet later, and which you have not yet considered in connexion with this formula. Beware of applying to these without caution results you have found true for ordinary arithmetical numbers.

21. In the same way discuss the area of a square measuring $a+b$ each way. In future the units of length and area will not always be explicitly mentioned ; it will be assumed that they correspond, so that if the lengths are supposed given in centimetres the unit of area will be the square centimetre, and so on.

The result of the discussion may be written

$$(a + b) \times (a + b) = a \times a + b \times b + a \times b \times 2,$$

$$\text{or } (a + b)(a + b) = aa + bb + ab2,$$

$$\text{or } (a + b)(a + b) = aa + bb + 2ab,$$

and the same caution is necessary about the applicability of this formula, and of all such formulas.

Ex. 5. In the same way discuss another expression for $(a+b+c) \times (a+b+c)$ by considering a square measuring $a+b+c$ in the side.

22. How would you calculate the price of 8 tons 7 cwt. of coal at 19s. a ton, and the price of 8 tons 18 cwt. at 19s. a ton? Direct calculation is possible, but a little foresight will often lighten the work.

In the first case the work is lightened by reckoning the price at £1 a ton and then subtracting 1s. a ton. In the

second case we may in addition reckon first the price of 9 tons and then subtract the price of 2 cwt. Further, we may make formulas useful for such cases, in which we use letters instead of numbers; so that any particular case may be treated by giving the letters numerical values instead of reasoning out the particular case for itself.

As in Arts. 19 and 20 such formulas are closely connected with formulas about rectangular areas, which will now be discussed.

23. Draw four rectangles measuring

$$\begin{array}{ll} a + b & \text{by } c - d, \\ a - b & \text{by } c - d, \\ a - b & \text{by } a - b, \\ a + b & \text{by } a - b, \end{array}$$

and express their areas in terms of rectangles having sides of lengths a b c d .

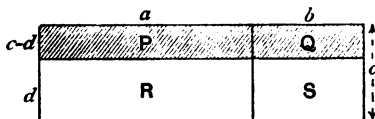


FIG. 6.

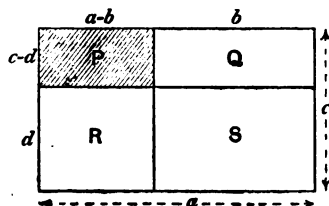


FIG. 7.

Figs. 6 and 7 need little additional explanation. In each the shaded part is the original rectangle. In each express the areas $a \times c$, $a \times d$, $b \times c$, $b \times d$ in terms of the rectangles P Q R S . By this means it is seen that in Fig. 6

$$(a + b) \times (c - d) = a \times c + b \times c - a \times d - b \times d,$$

or, more shortly,

$$(a + b)(c - d) = ac + bc - ad - bd,$$

and in Fig. 7

$$(a - b) \times (c - d) = a \times c + b \times d - a \times d - b \times c,$$

$$\text{or } (a - b)(c - d) = ac + bd - ad - bc.$$

Now draw Figs. 6 and 7 over again, but making $c = a$ and $d = b$. It results that

$$(a + b) \times (a - b) = a \times a - b \times b,$$

or $(a + b)(a - b) = aa - bb,$

and $(a - b) \times (a + b) = a \times a + b \times b - 2 \times a \times b,$

or $(a - b)(a + b) = aa + bb - 2ab.$

Ex. 6. Calculate to three significant figures 997×1014 and 997×997 .

Is there any limitation on the values that a , b , c , d may have?— a must be greater than b , and c greater than d , to make the rectangles possible.

Note. The notation of the index is better not introduced at this stage, not until the need is felt of shortening the writing; and when that need is felt, the pupils may with great advantage be asked to invent contractions.

EXERCISES.

7. Draw any closed curve on squared paper, and find its area by counting squares.

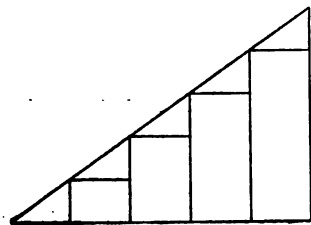


FIG. 8.

8. A triangle ABC , right-angled at B , has side AB 6 inches long and side BC 4 inches long. AB is divided into a number of equal parts, on which rectangles are constructed as in Fig. 8. Find the sum of the areas of the rectangles (i) when 9 in number (AB divided into 10 parts), (ii) when 99 in number (AB divided into 100 parts).

In each case find the difference between this sum and the area of the triangle.

9. Draw a straight line XQ , 9 inches long, and mark the position of a point Z , $1\frac{1}{2}$ inches distant from the line and near the middle of the line. Draw a straight line AB on tracing-paper; mark a point N on this line $2\frac{1}{2}$ inches from A . Move the tracing-paper about so that AB always passes through Z , and A is always on the line XQ . Find in this way the locus of N ; that is, for each position of A determine the position of N by pricking through the tracing-paper, and then draw a smooth curve through the points thus found.

Give reasons for thinking that the curve would never meet XQ .

10. Draw the field sketched in Fig. 9 (the lengths given being in yards) to a scale of 1 inch to 50 yards, and so find its area.

An ancient Egyptian formula for the area of a four-sided field is 'the mean of one pair of opposite sides multiplied by the mean of the other pair'. How much per cent. is this wrong in the present case? Name any case in which this formula is true, or give a simple reason why it cannot always be true.

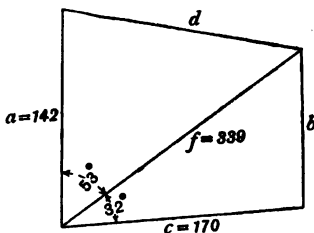


FIG. 9.

11. If a square has each side 3.4 inches long, what is its area?

If two boys were to calculate the area, one taking the length of a side as $(3.4 + x)$ inches, and the other taking it as $(3.4 - x)$ inches, what should be the difference between their results? And what (to the nearest tenth of a square inch) is this difference if $x = 0.04$?

12. If the floor of a storehouse is capable of bearing the weight of $4\frac{1}{2}$ cwt. to the superficial foot, how many high could you, with safety, stow cases measuring $2\frac{1}{2}$ feet by $1\frac{1}{2}$ feet, and weighing 1 cwt. 1 qr. 13 lb. each?

13. In order to determine the cross-sectional area of a river channel at a certain place a series of soundings are taken, and the results are given in the table below:—

Distance of sounding from right bank in feet . .	0	10	14	18	22	30	34	40
Depth of sounding in feet	8	17	20	23	17	10	7	0

Plot a section of the channel on squared paper, and compute the area of the cross-section in square feet.

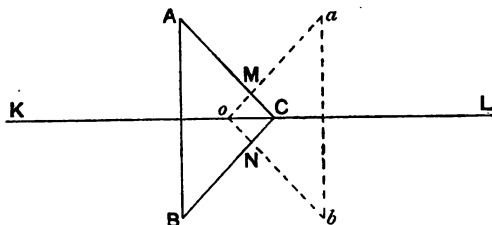


FIG. 10.

14. Draw a right-angled isosceles triangle ABC with hypotenuse

(that is, the side opposite the right angle) AB 10 cm., and a line KCL of indefinite length at right angles to AB . On tracing-paper make a copy oab of ABC and place it as shown in the rough sketch in Fig. 10. Now bring o to C , and starting from this point slide the tracing-paper to the left till ab coincides with AB , o always being on KL , and ab perpendicular to KL . If M and N are the points of intersection of the two pairs of sides, find from measurements (i) the area of $CMoN$ when o has moved 5 cm., (ii) the rate of movement of M if o moves uniformly at 1 cm. per sec.

15. Find the area of a regular hexagon whose side is 1 inch.

(A regular hexagon has six sides all equal and has each of its six angles 120° .)

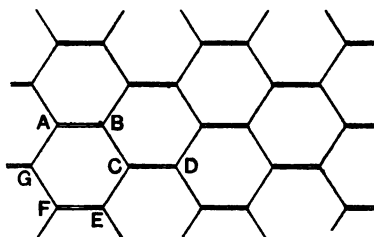


FIG. 11.

A piece of wire netting consists of hexagonal meshes an inch in the side (see the rough sketch in Fig. 11). Each horizontal side (such as AB or CD) of a mesh is formed of two wires, and each of the other sides, such as BC or CE , is formed of one wire.

Find, approximately, the length of wire contained in a square foot of the netting.

16. Construct a triangle ABC , given $A = 90^\circ$, $AB = 7.6$ cm., $AC = 5.7$ cm. On BC , CA , AB as bases, and exterior to the triangle, describe squares K , L , M . Given that there is a simple relation between the areas of K , L , M , find it by measurement and calculation.

Repeat with equilateral triangles K , L , M , instead of squares.

17. A recreation ground was surveyed by running a line through it. From the line offsets to the fence were taken at right angles at regular intervals of 50 links, the length in links of the successive offsets on one side of the line being 30 (at the starting-point), 37, 61, 45, 72, 80, 40, 60, 69, 48, 19. Give, as the decimal of an acre to the nearest hundredth, the area included between the line run through and the fence. 100 links = 1 chain. 10 square chains = 1 acre.

18. In a chemical experiment I require to dissolve one gram of copper in acid. I have a reel of copper wire 100 metres long, which weighs $3\frac{1}{2}$ kilograms. What length of wire must I cut off for the purpose? Mark off the length in question, writing your numerical answer below the measured length, calculated to the degree of approximation attainable in measuring.

19. The population curves, 1, 2 and 3 (in Fig. 12), are those of England, Ireland, and Scotland respectively. What was the population of each of those countries in 1841? When was the population of Ireland decreasing most rapidly, and that of England

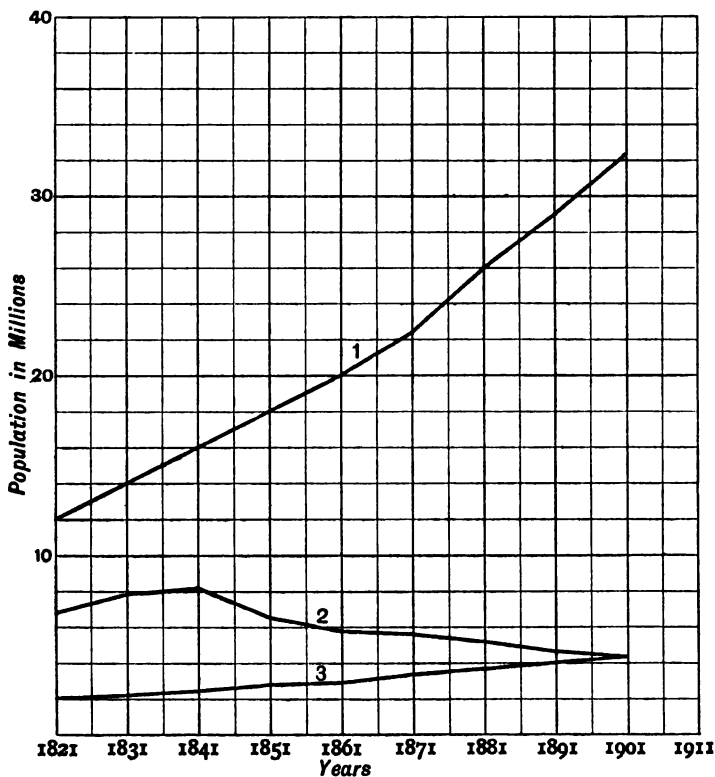


FIG. 12.

increasing most rapidly? Forecast from the curve the population of England in 1911.

(*Examples of reading:* The population of Scotland in 1831 was 2 millions, that of Ireland in 1821 was 6.8 millions, and that of England in 1871 was 22.5 millions.)

20. The Navy expenditure is given below for certain years. Plot these on a diagram and draw a broken line through the points. Draw also a straight line to lie as near this broken line as possible, and taking this straight line to indicate what the expenditure would be in 1905-6, and in 1906-7, if there was no change of plans, estimate the expenditure for each of these years. If the population may be estimated as 44 millions in 1906-7, what would the cost per head be in that year?

	£
1897-8	20,848,863
1898-9	23,880,875
1899-1900	25,731,220
1900-1	29,998,529
1901-2	30,981,315
1902-3	31,003,977
1903-4	35,709,477
1904-5	36,889,500

21. One man walks up a street 600 yards long at the rate of 200 yards in three minutes, but stops for five minutes in a shop half-way up the street. Another man, starting from the same end as the first, but seven minutes later, walks after him at the rate of 100 yards per minute. Draw lines on the squared paper which will indicate their respective distances from the end of the street at given times, using 1 inch to represent two minutes, and 1 inch to represent 100 yards.

Hence, or otherwise, find where and when the second man will overtake the first.

CHAPTER IV

VOLUMES

1. How shall we measure the air-space of the schoolroom ; how much oil an oil-drum holds ; how much coal can be mined from a given seam ; or how much water a swimming-bath or a reservoir holds ? What sort of unit do we need for these ?

Let us begin with the schoolroom.

Our unit must be a piece of space, such as a cubic metre, or a cubic foot. These are suitable for measuring the volume of a room. For smaller volumes the litre, gallon, cubic inch, cubic centimetre, &c., are useful.

Make a decimetre cube and an inch cube of paper by cutting out suitable pieces of paper, folding them, and pasting them together (see Fig. 1) ; leave margins for the pasting. If you can get pasteboard make a foot cube from it. In a corner of the room mark with chalk how a metre cube would stand.

Measure in metres the length, breadth, and height of the room you are in, which we will suppose rectangular. Suppose them to be 18 metres, 10 metres, and 4 metres, neglecting fractions for the present. (The teacher will of course use the actual dimensions in place of these.) If you covered

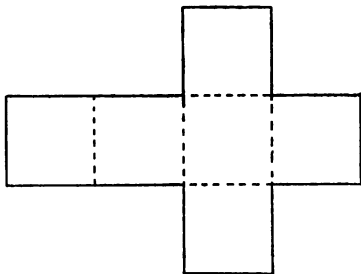


Fig. 1.

the floor with a layer of metre cubes, how many would there be, and what would be their volume?—180 cubes, and their volume 180 cubic metres.

How many layers would fill the room, how many cubes are there then, and what is the volume or air-space of the room?—4 layers, 520 cubes, and the air-space is 520 cubic metres, or, in terms of the measured dimensions, $18 \times 10 \times 4$ cubic metres.

The building bricks that children use, or clay or plasticine shaped into cubes, should be used as models.

2. Now measure the room more accurately, and suppose we find the dimensions in metres 18·6, 10·2, and 4·4. What additions must be made to the volume calculated?

With the building bricks a model, which is not to scale, may be made by increasing each dimension by a unit, or a model to scale may be made by using pieces of board along with the bricks. Either model shows that we can fill up the space left by—

(a) three slabs measuring (in metres) $18 \times 10 \times 0·4$, $10 \times 4 \times 0·6$, $4 \times 18 \times 0·2$;

(b) three pillars measuring $4 \times 0·6 \times 0·2$, $18 \times 0·2 \times 0·4$, $10 \times 0·4 \times 0·6$;

(c) a little block measuring $0·6 \times 0·2 \times 0·4$.

What is the volume of a slab, say the horizontal one? How many metre cubes would ten such slabs together make?—They would make $18 \times 10 \times 4$ cubes, so that one slab has a volume of $\frac{18 \times 10 \times 4}{10}$, or $18 \times 10 \times 0·4$ cubic metres.

What is the volume of a pillar, say the vertical pillar? How many such pillars would you put together to be able to cut them up into metre cubes?

And how many copies of the little block would you put together to be able to cut the figure so made into metre cubes?

We can now calculate the volume of each piece, and we see that to find the total volume we carry out operations that could be used to give the product $18·6 \times 10·2 \times 4·4$, or

610·868. Give separately the volumes of the original piece, the slabs, the pillars, and the little block. They are

	c. m.
Original piece	520
Slabs	86·4
Pillars	8·92
Little block.	0·048
	610·868

and, assuming the dimensions of the room to have been measured to the nearest decimetre, we give as an appropriate result for the air-space 610 cubic metres. What is the order of importance of the various volumes you have given? And how many of them affect the result if it is wanted to one significant figure? If to two significant figures?

If you had begun with the decimetre and cubic decimetre as units, what would you have had for dimensions and volume of the room? And what is this volume expressed in cubic metres?

3. Give a formula for the air-space in terms of the dimensions of the room. The air-space and the dimensions being expressed in corresponding units, we have

$$\text{volume} = \text{length} \times \text{breadth} \times \text{height},$$

or, if we denote volume, length, breadth, and height by v, l, b, h ,

$$v = l \times b \times h,$$

or

$$v = lbh.$$

Need the length, &c., be whole numbers or involve only decimals of a particular kind?—No; for decimals to any number of places we can proceed by using successive units, smaller and smaller; or we can begin with a unit small enough for the smallest volume that occurs; and in either case we arrive at the formula

$$v = l \times b \times h.$$

Are vulgar fractions also included? Vulgar fractions could be considered for themselves in the same way as decimals, but any vulgar fraction can be expressed decimally

—by a terminated decimal—to any degree of accuracy we choose; and therefore, to any degree of accuracy we choose, vulgar fractions are already included.

Later we shall meet with numbers that cannot be expressed with absolute accuracy by either decimal or vulgar fractions; for instance, the length in inches of the diagonal of an inch square (that is, the line joining opposite corners). But these numbers also can be expressed decimally to any degree of accuracy we choose. So that for any rectangular piece of space of length l , breadth b , and height h , the volume (in the corresponding unit) is given by

$$v = l \times b \times h.$$

Ex. 1. In Huxley's opinion the air in a room ought to be renewed at the rate of 60,000 litres an hour for every person present. On this reckoning, how often would complete renewal of the air be necessary in the room you are in?

The standard gallon measure.

4. Get a standard gallon measure, and try to determine its capacity in cubic inches. How will you set about it?—We could try how many inch cubes can be packed in. Or we could try how many go to the bottom layer and how many layers there are.

And what is the relation between the number of cubes in a layer and the area of the bottom of the gallon measure?—The number of cubes in a layer is the number of inch squares that could be marked out on the bottom.

Measure the inside diameter of the gallon measure (7.1 inches), draw a circle of this diameter to represent the bottom, and mark out on it as many inch squares as you can.

In packing-in the inch cubes you left some corners unfilled. What will you do with these?—We need smaller units. These might be obtained by supposing an inch cube sliced into ten equal slabs for the first smaller unit, a slab sliced into ten equal pillars for the second unit, and a pillar cut into ten little cubes measuring 0.1 inch each way for the third unit.

Of these three kinds of smaller units how many kinds do you use to complete approximately the bottom layer, and how are they related to areas on the bottom of the vessel? On your drawing of the bottom of the vessel show the bases of all the units used for the bottom layer.

Count up the area thus covered on the drawing, and the volume of the units that would form the first layer in the vessel.

5. Measure the inside height of the vessel (7·1 inches). How many layers like the bottom layer could you pack in, and what volume would they occupy?—Seven layers, and their volume in cubic inches would be seven times the area in square inches of the bottom of the vessel. And what further layer do you need?—A thin layer, 0·1 of the thickness of the bottom layer, so that its volume is 0·1 of the volume of the bottom layer.

And the whole volume?—It is in cubic inches 7·1 times the area in square inches of the bottom. Compare this result with that for the schoolroom. In each case, all the quantities being expressed in corresponding units,

the volume = the area of the base $\times$ the height.

The floor of a room is said to be horizontal and the walls to be vertical. When the gallon measure stands on a table its base is also horizontal and its sides vertical. And in our discussion of these cases the shape of the floor and the shape of the bottom of the gallon measure were of no consequence, so that we know that in such cases, whatever the shape of the base,

volume = area of base $\times$ height.

6. Look up arithmetical tables to find how many cubic inches a gallon is equal to, and compare with your former result. Can you explain why they differ?—We did not completely fill up the vessel with our solid units, and the units of area marked out on the circle that represented the bottom did not completely fill the circle.

Consider the parts within the circle not accounted for, judge by eye how many squares measuring 0·1 by 0·1 they are equal to, correct your former result, and compare again with the true result*.

Ex. 2. Each member of the class draws a circle at random. Then he measures its circumference (that is, the length of the

* There is another source of error in the approximation 7·1 inches used for diameter and height.

curve) by stepping round it with dividers. He also measures the diameter, and calculates the quotient

$$\frac{\text{circumference}}{\text{diameter.}}$$

Then all the results from the different circles are collected and tabulated in three columns, under the heads—

Diameter in cm.		Circumference in cm.		Circumference ÷ diameter.
--------------------	--	-------------------------	--	------------------------------

Is there anything noticeable about the results? Would the third column be much affected if the measurements were taken in inches?

Ex. 3. Take a coin and measure its circumference by rolling it along a graduated ruler. Measure its diameter and calculate
circumference ÷ diameter.

Ex. 4. Each pupil again draws a circle of any size on squared paper, and draws a square on a radius of the circle. Then he finds the area of the circle and of the square by counting squares, judging by eye what to allow for the squares partly included. Then he calculates

$$\text{area of circle} \div \text{area of square on radius.}$$

All the results are collected and tabulated under the heads—

Area of circle in sq. cm.		Area of square on radius in sq. cm.		Area of circle ÷ area of square.
------------------------------	--	--	--	-------------------------------------

Is anything noticeable about the results?—It is noticed that the numbers in the last column are pretty nearly the same, and pretty nearly the same as in the last column of the preceding table. Would you expect any of these results to be quite accurate?—No; the measurement of a length is not likely to be more exact than to the nearest millimetre, or at the outside to the nearest tenth of a millimetre; the eye is no great judge of the area of the squares partly included; each line has some thickness; and in stepping the dividers along the circumference it is difficult to step fairly on the line.

7. Further developments of mathematics show that if all the operations mentioned in Exercise 4 could be accurately carried out, all the numbers in the third columns would be the same number, and they show how this number can be calculated to as many decimal places as you please. It has been calculated to a great many places, but the value 3.1416 is as accurate as you are likely to need, and for most purposes 3.14 is near enough. This number occurs so often that (to save writing)

a contraction is used for it, namely the Greek letter π ; so that we may say the circumference of a circle is 3·14 times the diameter, or (if we need to be more accurate) 3·1416 times the diameter, or we can say it is π times the diameter, leaving over the decision which of these values is to be used for π , or whether a still more accurate value is to be used.

And, calling the radius of the circle r , we may write down the area of the circle in various ways, a few of which are:—

3·14 times the square on the radius,

$3\cdot1416 \times \text{radius} \times \text{radius}$,

$\pi \times r \times r$,

$3\cdot14rr$,

πrr .

8. *Averages.* There were various reasons why the values in the last columns of the tables in Exercises 2 and 4 were not likely to be accurate. Are the values likely to be too great or too little?—They might be either, some are likely to be too great and others too little.

If you arrange the values in order of size, greatest first, those at the beginning are likely to be too great, those at the end too small, while one from the middle of the list is likely to be nearer the true value than one from either end. How will you select a likely number from the list? We might choose the middle number. Or we might take a number midway between the two extremes. Or we might (and this is the usual way) add together all the values and divide this sum by the number of these values. The result of this is called the average of the values. Calculate the average of the values in the third column of each table and compare it with 3·14.

Ex. 5. In a company of soldiers

49 were 68 inches in height

33 " 69 "

14 " 70 "

10 " 71 "

1 " 73 "

What was the average height?

If there were A soldiers of height a , B of height b , C of height c , and D of height d , what would be the average height?

Ex. 6. A boy made in four successive examinations the marks 73, 84, 90, 100. What was his average mark? After the second examination the boy calculated his average as $\frac{73+84}{2} = 78.5$, after the third examination he reckoned his average as $\frac{78.5+90}{2} = 84.2$, after the fourth he reckoned it $\frac{84.2+100}{2} = 92.1$. Criticize his method, showing to which examinations he gave too much weight and to which he gave too little.

9. We have seen that a circle of radius r has an area (in corresponding units) of $3.14 \times r \times r$. Use this expression to calculate the area of the bottom of the standard gallon measure, and the capacity of the vessel in cubic inches, and compare with your previous results. Also give an expression for the capacity of such a vessel whose height is h and the radius of whose base is r .

Ex. 7. Take a standard litre, measure its height (17.2 cm.) and its diameter (8.6 cm.), and find in two ways its capacity in cubic centimetres. Compare with the value given in arithmetical tables.

*Volume of a reservoir.**

10. A reservoir has been made by damming up a valley. The bottom may be taken as level ground, 26 hectares in area. The area of the water surface as the water rises is as follows:—

Depth in metres	...	5	10	15	20	25	30
Area of water surface	}	...	37	49	66	85	104
in hectares							
							120

and the reservoir is full when the depth is 30 metres. How will you determine the quantity of water it will hold? Can you divide the water approximately into horizontal layers bounded by vertical sides?

Suppose a wall built round the 26 hectares of level bottom, carried up 5 metres, and the space behind filled

* The method of finding limits between which a quantity lies, discussed in articles 10–12, should be first introduced in connexion with some simpler problem than the reservoir. For instance, the class might, for the area of a circle drawn on squared paper, get an upper limit by counting the squares partially included, and a lower limit by leaving out of count the partially included squares.

up flush with the wall. Then when the water stands at 5 metres it is of a shape that we know how to deal with; it has the same shape as the gallon measure with a horizontal bottom and vertical sides, a shape called cylindrical. What is the volume, the height being 5 metres and the area of the base 26 hectares? Is it 26×5 with any unit?—No, because metres and hectares are not corresponding units; we had better first express the 26 hectares as 260,000 square metres. Then we know the volume is in cubic metres $5 \times 260,000$ or 1,300,000.

11. If now another wall is built round the water's edge while it stands at a depth of 5 metres, and then the reservoir filled to a depth of 10 metres, we shall have another cylindrical piece of water of volume 1,850,000 cubic metres. Treating the other layers in the same way, we have the volumes of the six layers, in cubic metres,

1,300,000
1,850,000
2,450,000
3,800,000
4,250,000
5,200,000

or in all 18,350,000 cubic metres. We know that this is not exact, that it is less than the true volume, but we do not know how much less. Could you find a result that is more than the true volume, and thus have some idea how far wrong our results may be?

We could obtain cylindrical layers by cutting away the banks of the reservoir instead of filling up with walls. Thus we begin at the line of the edge the water has when 5 metres deep and cut down all round to the bottom level; this gives, when the reservoir is filled again to 5 metres, a layer 37 hectares in area and 5 metres in thickness. Treating each layer in the same way, we have now for the six layers, in cubic metres,

1,850,000
2,450,000
3,800,000
4,250,000
5,200,000
6,000,000

or in all 28,050,000 cubic metres. And we know the volume of the actual reservoir to be less than this. Comparing the two results, we may say that the volume is about 20,000,000 cubic metres.

Could this work have been shortened?—As each of the areas had to be multiplied by 5 and these products added, we could have added the areas and then multiplied by 5.

12. If the area of the water surface of a reservoir at different levels is given by

Level of water in metres	0	p	$2p$	$3p$	$4p$	$5p$	$6p$
Area of water surface in sq. metres	a	b	c	d	e	f	g

give two approximate expressions for the volume of water. One of these is, in cubic metres,

$$\begin{aligned}
 & p \times a + p \times b + p \times c + p \times d + p \times e + p \times f, \\
 \text{or} & \quad pa + pb + pc + pd + pe + pf, \\
 \text{or} & \quad p \times (a + b + c + d + e + f), \\
 \text{or} & \quad p(a + b + c + d + e + f),
 \end{aligned}$$

and the other, written in the shortest of these forms, $p(b + c + d + e + f + g)$.

Is there any limitation on the numbers that these letters may represent? The discussion assumed the area of the water surface to increase as the water rose, so that $a b c d e f g$ must be in order of increasing magnitude, that is, each is greater than any preceding one.

13. Suppose there is a series of boxes, the bottom of each measuring 80 cm. by 60 cm., while the heights differ. Calculate the volume for various heights, and show in a graph how the volume varies as the height changes.

Suppose another series of boxes all of the same height, 70 cm., and with the bottoms square and of varying size. Calculate and show in a graph how the volume varies as the side of the square bottom varies.

Suppose a third series of boxes, all cubes, and varying in size. Show in a graph how the volume varies as the length of the edge varies.

In the first of these three cases the volume depends on,

or varies with, the height of the box, or, in mathematical language, is a function of the height of the box. In the second the volume depends on, or varies with, or is a function of, the length of a side of the square bottom. In the third the volume depends on, or varies with, or is a function of, the length of an edge of the cubical box.

14. One man *A* lends another *B* £10,000, and *B* agrees to pay at the end of every year for the use of this money £4 for every £100 he has borrowed, or, in the usual semi-Latin phrase, 4 per cent. How much does he pay annually on the loan of £10,000?—He pays £400.

If *A* and *B* agree that this £400 need not be paid over but considered as an additional loan, what does *B* owe at the end of the year?—He owes £10,400. For the second year *B* has now to pay on his loan of £10,400. What is the payment on this; and, if this also is added to the loan, what is the amount of the loan at the end of the second year?

On this plan, by what do you multiply the original loan to find the amount of the loan at the end of the first year? And by what do you multiply the amount of the loan at the end of the first year to get the amount at the end of the second?—In both cases the multiplier is 1·04.

On this plan, to what does a loan of £1 increase in a year?—It increases to £1·04. And in two years?—£1·04 × 1·04. And in three, or four, or five years?—To £1·04 × 1·04 × 1·04, or £1·04 × 1·04 × 1·04 × 1·04, or £1·04 × 1·04 × 1·04 × 1·04 × 1·04.

These expressions become tedious to write. The pupils should be encouraged to express this tedium, and then asked for suggestions of shorter ways of writing the expressions. The usual contraction is not immediately necessary, and the work may go on for a while without it, or with the help of contractions invented by the pupils.

Now calculate what a loan of £1 would increase to in four years, for different rates of payment for its use, say for 1, 2, 3, . . . 10 per cent. Tabulate these, and also show in a graph how the total to which the loan increases depends on the rate per cent., that is, show the total for 4 years as a function of the rate per cent.

Give an expression for the amount to which a loan of £1 will increase in four years at r per cent. It is

$$£\left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right).$$

What will it increase to in ten years?

If in the expression for the amount in four years the brackets were obliterated, what would the expression mean? Compare this meaning and the original meaning when $r = 10$ and when $r = 50$.

EXERCISES.

8. Measure, in centimetres, a rectangular block of wood, and calculate its volume.

Weigh it and find how much a cubic decimetre would weigh.

9. A mahogany pattern, made from wood weighing 762 grams per cubic decimetre, weighed 2.84 kilograms. A gun-metal casting of the same size and shape weighed 30.8 kilograms, the gun-metal employed weighing 8.46 kilograms per cubic decimetre. Show that there must be a cavity in the casting, and find what weight of metal would fill it. All the numbers are given to three significant figures.

Take the weight of the wood as a grams per cubic decimetre, of the gun-metal as b kilograms per cubic decimetre, of the pattern as c kilograms, and of the casting as d kilograms, and give an expression for the weight of metal that would fill the cavity.

10. Draw any rectangle and suppose it to represent the bottom of a cistern on a scale of 1 inch to 1 foot. If 200 gallons of water are run into the cistern, to what level will they fill it? A cubic foot is 6.23 gallons.

11. A room 460 square feet in area is to be floored with wood blocks, the flooring to be 2 inches thick. What will the blocks cost at 3s. 6d. per cubic foot?

A room p square feet in area is to be floored with wood blocks, the flooring to be q inches thick. What will the blocks cost at r shillings a cubic foot?

12. The ends of a workshop are 10 metres wide, the height at the ridge of the roof is 8 metres and at the eaves 5.2 metres. The workshop is 18 metres long. What is the floor area in square metres, and the air-space in cubic metres?

13. A ton of lead is rolled into a sheet $\frac{1}{8}$ -inch thick. Determine the area of the sheet in square feet, given that 1 cubic foot of lead weighs 712 lb.

A ton of lead is rolled into a sheet m inches thick. Determine

the area of the sheet in square feet, supposing a cubic foot of lead weighs n pounds.

14. The royalty paid by a mining company for a horizontal seam of coal is £100 per acre per foot thick. Find how much this amounts to per ton, a cubic foot of the coal weighing about 70 lb.

15. Find, to the nearest penny, the value of 100 yards of copper wire, of which the section is circular, of diameter $\frac{1}{8}$ of an inch. (1 cubic foot of copper weighs 8,570 oz.; the value of 1 ton of copper is £78 15s.)

Supposing 1 ton of copper worth a pounds sterling, and 1 cubic foot of copper to weigh b ounces, find an expression in shillings for the value of m yards of copper wire of diameter n inches.

16. A rectangular subway is to be made b feet wide and c feet high. The earth weighs x pounds per cubic foot. How many tons of earth will be removed in making l feet of the subway?

If air is driven in at one end of the subway at the rate of k cubic feet per minute, what will be the speed of the air along the subway?

Give numerical answers for the case in which $b = 9$, $c = 8$, $x = 70$, $l = 1$, $k = 2,000$.

17. Fig. 2 shows the plan of a church, the east end terminating in a semi-circle. Find the area of the floor.

The average height of the church is 60 feet, and it is estimated that for health every individual requires a supply of fresh air at the rate of 2,000 cubic feet per hour. A service lasts $1\frac{1}{2}$ hours; at the beginning the air is fresh, but there is no renewal of air during the service. How many will the church accommodate without injury to health?

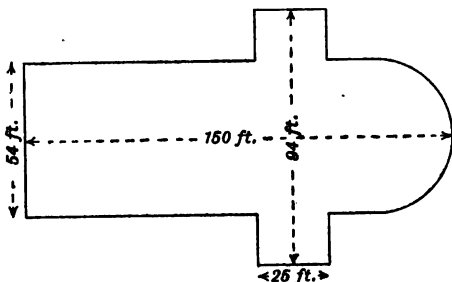


FIG. 2.

18. A cubic block of wood, a inches in the edge, and weighing 0.8 of an equal volume of water, floats in water. Salt is gradually dissolved in the water till the brine is 1.2 times as heavy as water. How high will the block rise from its initial position? Assume that the weight of the block is equal to the weight of the liquid displaced.

19. Find the volume of water a reservoir will hold, given that its greatest depth is 30 feet; the areas of the surface of the water A ,

at heights h above the lowest point of the bottom, were measured and found to be :—

A in sq. yds.	0	2100	8200	13000	15400	20000	25600
h in feet . .	0	5	10	15	20	25	30

Explain the method of calculation you adopt.

20. If a square post has to be cut from a cylindrical tree trunk, find roughly what percentage of the wood is wasted.

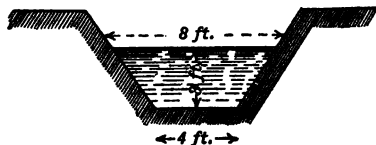


FIG. 3.

21. Fig. 3 shows the section of a small watercourse. The surface velocity is 20 inches per second, and the mean velocity = the surface velocity $\times 0.8$. Find the discharge in gallons per minute.

22. The population of a town increases uniformly, and in each period of three years the increase is 20 per cent. of the population at the beginning of that period. If the population was 90,000 in January 1903, what will it be in January 1906 and 1912, and what was it in January 1897?

By drawing a graph, find the population approximately in January of each year from 1900 to 1905.

23. The table below shows the distances (in miles) from London of certain stations, and the times of two trains, one up and one down. Suppose each run to be made at constant speed, and show by a graph the distance of each train from London at any time, using 1 inch to represent 20 miles, and 3 inches to represent an hour.

Distances {	London	4.30 p.m.	↑ 7.0 p.m.
	5½ Willesden, arrive .	4.38 "	(No intermediate stop.)
	" , depart .	4.42 "	
	66 Northampton, arrive	↓ 5.50 "	
	" , depart	5.54 "	
	113 Birmingham	7.0 "	5.0 p.m.

At what point do they pass one another, and how far is each from London at 5.30? Which of the three runs made by the stopping train is the fastest?

CHAPTER V

TO MAKE A COPY OF A MAP.

1. LET us now discuss the problem how to make a copy of the map shown in Fig. 1.* How will you proceed? Let us begin with Kingston. Where will you place it?—Any-

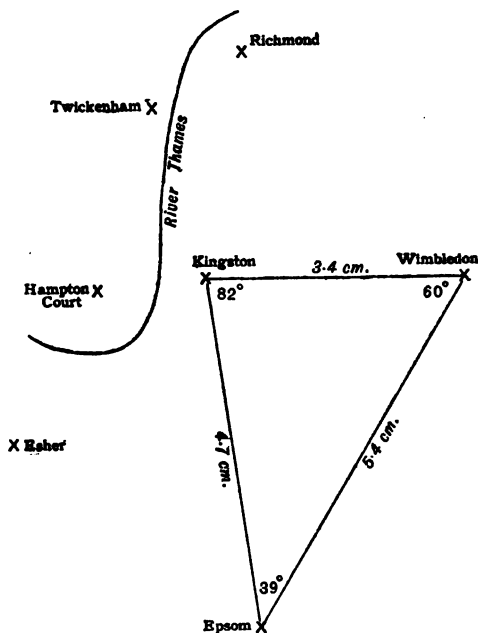


FIG. 1.

* A local map should be used instead of that given, and the places on it chosen so that the distances to measure are greater.

where. Put it then at K (Fig. 2). And where will you put Wimbledon? What condition must it satisfy?—It must be at the same distance from Wimbledon as on the given map, namely, 3.4 cm., and anywhere at that distance will do.

Put it down then at W , anywhere on a circle with centre K and radius 3.4 cm.; and consider Epsom. What do you know of its position?—Epsom is 4.7 cm. from Kingston, so that it must lie somewhere on a circle with centre K and radius 4.7 cm. Will it do anywhere at random on this circle?—It is 5.4 cm. from

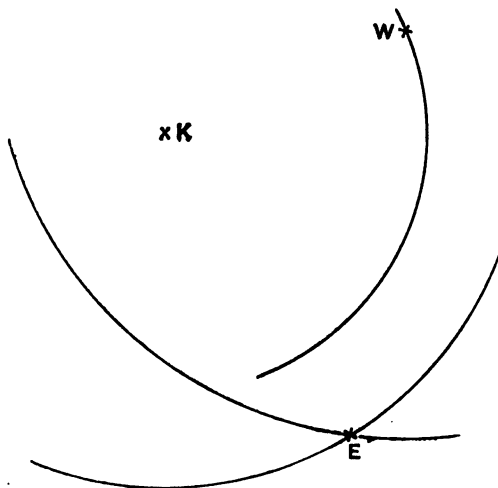


FIG. 2.

Wimbledon; which gives another circle on which Epsom must lie.

Do these two circles fix a certain point where Epsom must lie? How will you decide between the two points in which the circles cut, or are you at liberty to choose one of them at random?

The pupils will easily see that only one of the points will do, but not so easily how to distinguish it. The usual convention that the top of the map is the north, so that Wimbledon is east of Kingston, will enable them to place their copies with W east of K , and to distinguish Epsom as the intersection

that lies south of the line KW . Or without using the points of the compass, the pupils may be led to notice that a man going from Kingston to Wimbledon has Epsom on the right.

Having now K , W , and E on the map, how will you place Esher?—By measuring its distances from two of the places already marked and proceeding as before. And the remaining places?

2. Now count up how many measurements you have made to copy the map.—We had

For Kingston	0	measurements
„ Wimbledon	1	„
„ Epsom	2	„
„ Esher	2	„
„ Richmond	2	„
„ Twickenham	2	„
„ Hampton Court	2	„

a total of 11 measurements for 7 places.

If there had been 20 places on the map, how many measurements would have been needed?—That means 18 more places at 2 measurements each, or in all $11 + 26$, that is, 37 measurements. In the whole number of measurements how many less are there than 2 each for all 20 places?—8 less, 2 being saved on the first place put down, and 1 on the second place put down.

How many measurements are wanted for a map of N places?—2 for each, except for the 3 saved on the first two places, that is, $2 \times N - 3$ measurements.

How would you copy the river? How would you fix a number of points on it?—We could fix as many points as we like in the same way as we fixed the places, and then draw the river freehand through them.

3. Return now to consider the three places K , W , E . Say again how they were fixed, and how many conditions or measurements were needed to fix them. Would it make any difference if we began by putting down W first of all instead of K ? Or if we had taken E second instead of W ?

Can you suggest anything else we could have measured in place of the three lengths KW , KE , WE ? Could you make use of your protractor?—We could measure the

angles at K , W , and E . Do so, and measure the corresponding angles on your copy of the map.

We have now 6 measurements, 3 lengths, and 3 angles, and we know that the 3 lengths are enough to enable us to make a map of K , W , E . If we had only 2 lengths, KW and KE , what more is needed? Find by trial.—An angle in addition will do.

Does it matter which?—The angle K will do; whether one of the other angles would do is not clear.

Measure the angle at K , and use it to draw the triangle KWE .

If you had only the length KW , could you make up with angles?—With the angles at K and W we could draw the triangle. Measure these angles and do so.

Suppose that you measure only one length KW , and for the rest measure angles only. Could you in this way copy the whole map? Do so.

Suppose again that you copy the whole map in this way, but by an error make KW of length 6 cm. instead of 8.4 cm. Draw the whole figure that you get in this way and compare it with the original map. Does it look like the original map?

4. *The Plane Table*.—A man takes a sheet of paper on a drawing-board to a high point in Wimbledon from which he can see a wide stretch of country. He fixes the board horizontal, and marks a spot W on the paper to represent Wimbledon. Then from W he draws lines pointing to Kingston, Epsom, &c., and labels them 'to Kingston', &c. He now removes to Kingston, where we must suppose him to go up in a captive balloon to command a wide stretch of country. On the line labelled 'to Kingston' he marks a point K to represent Kingston. Then he sets up the board so that the line KW points to Wimbledon, and draws from K lines to Epsom, &c., and labels them 'to Epsom', &c., as before. There are now two lines labelled 'to Epsom', and he marks Epsom at their point of intersection; and so with all the other places. In this way he makes a map of the district.

Select two high points in your neighbourhood, and use this method to make a map of the district.

A board made with certain attachments so that it can be levelled and fixed for accurate use in this way is called a plane table.

5. *Angles*.—In order to lay down *W* at the proper distance from *K*, you measured *KW* as 3.4 cm., and laid off this distance. Could you have laid off the distance without a centimetre scale?—Yes, by opening the compasses to reach from *K* to *W* and laying off this distance.

You measured the angles with your protractor, and laid them off with the protractor. Could you have done without a protractor? Lay down *K* and *W* without using a centimetre scale, and consider how you will make the angle *WKE* of the right size.—We could cut or fold a piece of paper (Fig. 3) so as to fit the angle at Kingston.

Then *K'* being laid on *K* and *K'P* along *KW*, *K'Q* gives the direction of the line *KE*.

Use this method to lay down Epsom, Richmond, &c., on your map, considering all the time whether you can in any way simplify the procedure.

By repetition of the cutting out of pieces of paper to fit the angles the pupils will discover, or be led to discover, that the pieces *A* and *B* (Fig. 4) of the scrap of paper used need not be cut away, that it is sufficient to lay *K'* on *K* of the original map and rule *K'P* and *K'Q* in line with the parts of *KW* and *KE* that project beyond. Then it is seen that

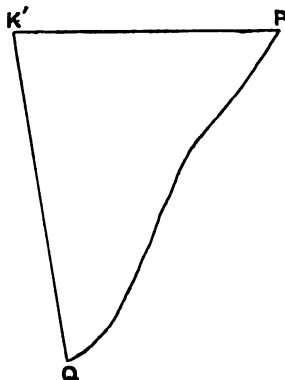


FIG. 3.

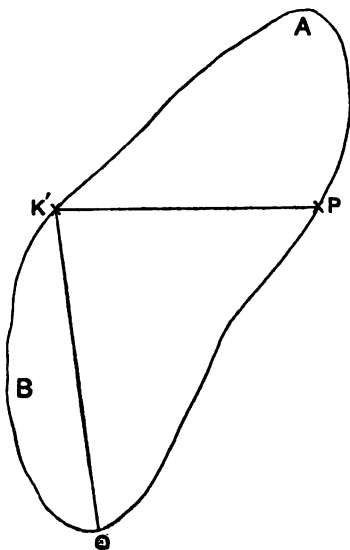


FIG. 4.

it is sufficient to mark the points $K' P Q$ without ruling any lines. The next step is that if we know the lengths $K'P$, $K'Q$, and PQ we can always place the points $K' P Q$ in proper relative position without using the scrap of paper at all. That is, P and Q are taken anywhere on KW and KE on the original map, KP is laid off of the proper length on the new map, and Q is found as the intersection of two circles, one with centre K and known radius KQ , and the other with centre P and known radius PQ . By carrying this out with as few operations as possible we arrive at the following construction.

6. To draw from Q (Fig. 5) a line making with QP an

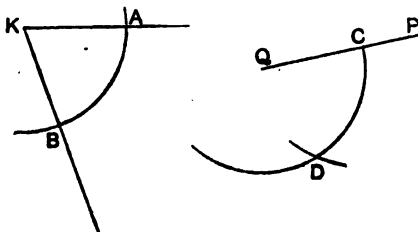


FIG. 5.

angle equal to the angle K ; with the point K as centre draw a circle cutting the two arms of the angle in A and B , and with Q as centre draw an equal circle cutting QP in C . Then take the length AB on the compasses and lay off this length as a chord CD in the second circle, and join DQ . DQP is an angle equal to K .

How many points D are there?

7. Could you bisect a given angle K (Fig. 6), that is, draw a line dividing it into two equal parts?—One way is to fold the legs KL and KM together. The crease line KN bisects the angle, since the two parts LKN and MKN fit when folded together, and are therefore equal.

Could you adapt the method just discovered for copying an angle to the bisection of an angle?—If we can choose A on KL , B on KM ,

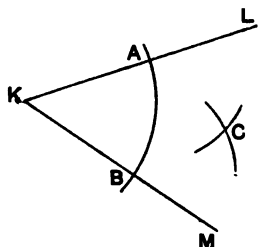


FIG. 6.

and C in the angle, so that $KA = KB$, $AC = BC$, and $KC = KC$, we shall have two triangles that fit together, so that the angles AKC and BKC will be equal. To do this, mark off KA and KB equal to one another; then with centres A and B and any convenient radius draw circles; either point where they meet will do for C .

Ex. 1. Take two points A and B 5 inches apart, and suppose these to represent two points on a straight coast, 1000 yards apart. Simultaneous observations are taken at A and B of a ship S as it sails past, there being four pairs of observations, as follows:—

1. $SAB = 96^\circ$, $SBA = 38^\circ$.
2. $SAB = 65^\circ$, $SBA = 30^\circ$.
3. $SAB = 35^\circ$, $SBA = 33^\circ$.
4. $SAB = 33^\circ$, $SBA = 81^\circ$.

Plot the 4 positions of the ship. Assuming that the ship's course is composed of two straight lines, find how near she approached the shore. Assuming that she sails equally near the wind on each tack, draw a line showing the direction of the wind.

Fixing a triangle.

8. It appears that there is a good deal of choice in the measurements that may be taken to make a copy of a map; and also that copying a map means fixing in succession a number of triangles, that is, drawing the triangle, or taking measurements enough to be able to draw the triangle. Let us then consider in what different ways we can fix a triangle.

We have seen that there are 6 things about a triangle that we can measure, namely, 3 sides and 3 angles. For shortness we will call the sides a b c and the angles A B C , the angle A being opposite the side a , and so on (Fig. 7).

Could you draw the triangle ABC from a knowledge of 1 side? From a knowledge of 2 sides? Of 3 sides?—The discussion of the map shows that 3 sides are enough, but not a less number; and knowing 3 sides we know all about the triangle, for we can draw it and measure the angles or anything else about it.

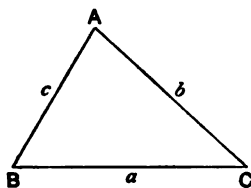


FIG. 7.

Make a triangle having $a = 7.2$ cm., $b = 8.0$ cm., $c = 9.8$ cm., and measure the angles. Try to make a triangle having $a = 6.2$ cm., $b = 8.0$ cm., $c = 1.6$ cm. What condition must the given lengths satisfy to make a triangle possible?—Any two sides together must be greater than the third.

9. If two triangles are made with the same values for a b c , can they be fitted together?—Lay down the side a (Fig. 8), then a knowledge of the length b tells us that A lies

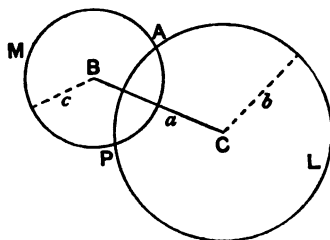


FIG. 8.

somewhere on a circle L with centre C and radius b , and the length c tells us that A lies on a circle M with centre B and radius c . So that A must lie at one of the two intersections of these circles, say at the point marked A . Suppose another triangle $A'B'C'$ made, having $B'C' = a$, $C'A' = b$, $A'B' = c$, and lay it on the other so that B' lies on B , and C' on C . Then, since $C'A' = b$, A' lies on a circle with centre C and radius b , that is, on the circle L , and in the same way it lies on the circle M ; so that it lies at A or at P . If we originally placed $A'B'C'$ so that A' lies on the same side of BC as A , then A' falls on A and the triangles fit. Or we could allow it to fall either way; and if A' falls at P , we know from a former discussion (chapter I) that folding along BC brings P on A , so that in this case also the triangles fit together.

10. You know that 2 sides won't fix a triangle. Could you in place of the third side take any other measurement or measurements?—We could measure an angle, or, if need be, 2 angles.

Try to make nine triangles in which (all the lengths being in centimetres):—

- | | | | |
|----|-------------|-------------|-------------------|
| 1. | $a = 6.9$, | $b = 9.4$, | $C = 50^\circ$. |
| 2. | $a = 6.9$, | $b = 9.4$, | $B = 50^\circ$. |
| 3. | $a = 9.4$, | $b = 6.9$, | $B = 50^\circ$. |
| 4. | $a = 9.4$, | $b = 4.3$, | $B = 50^\circ$. |
| 5. | $a = 9.4$, | $b = 4.3$, | $A = 50^\circ$. |
| 6. | $a = 9.4$, | $b = 4.3$, | $C = 10^\circ$. |
| 7. | $a = 9.4$, | $b = 4.3$, | $C = 150^\circ$. |
| 8. | $a = 9.4$, | $b = 4.3$, | $C = 200^\circ$. |
| 9. | $a = 6.9$, | $b = 9.4$, | $B = 130^\circ$. |

The pupils should attempt these constructions with little or no assistance. Half-successful and unsuccessful attempts are of value for the following discussion.

For the fixing of a triangle by measurement of 2 sides and an angle or angles we may take as an angle to be measured the one between the 2 measured sides, or one not between, that is, one opposite to one of the sides. Let us take the angle between the sides, and call the sides a and b ; then the angle is C . Are these 3 measurements enough to fix the triangle?—Yes, for we can make an angle C with the protractor, and measure the lengths a and b along its two legs.

If two triangles are made with the same values of a b C , will they fit together?—Yes, as we see by fitting the 2 angles together so that the 2 longer legs fall together, and the 2 shorter together.

Copy the triangle of Fig. 7 by means of its a b C , and without using a scale or protractor.

Can a triangle always be made from given values of a b C ?—The lengths of a and b do not matter. C must be less than 180° .

Draw two lines 3.2 cm. and 1.9 cm. long, and inclined to one another at 200° , and join the ends of the lines. Has this triangle an angle of 200° ?—No, the angle of the triangle is 160° at the corner where we made an angle of 200° (Fig. 9).

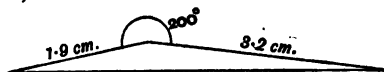


FIG. 9.

11. We know that 2 sides alone are not enough to fix a triangle. Suppose, for instance, that we know $a = 6$ cm.

$b = 10$ cm., but nothing more. Let us lay down $BC = 6$ cm., Then what do you know of the position of A ?—That it

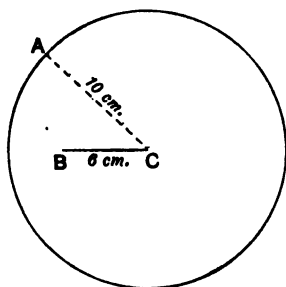


FIG. 10.

lies on a circle with centre C and radius 10 cm. (Fig. 10, a sketch drawn on a reduced scale).

If now you are given also that $A = 28^\circ$, how will you find the position of A ? Try drawing the triangle with A in various positions on the circle. Or better, by drawing the triangle in various positions and measuring the angles A and C draw a graph giving the angle A as a function of the angle C

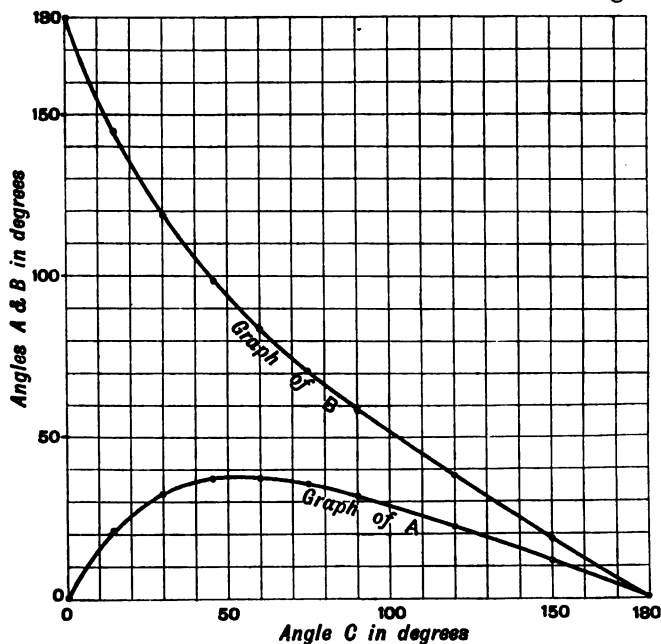


FIG. 11.

(Fig. 11). From this graph read off the value of C that corresponds to $A = 23^\circ$.—There are two values, namely, $C = 18^\circ$ and $C = 115^\circ$ approximately.

Make a triangle having $a = 6$ cm., $b = 10$ cm., $C = 18^\circ$, and another having $a = 6$ cm., $b = 10$ cm., $C = 115^\circ$, and measure the angle A in each.

What is the greatest value A can have with the given values of a and b ?—The graph shows it to be about $37^\circ.5$.*

Make the triangle having $A = 37^\circ.5$.—The graph gives $C = 48^\circ$, and we make the triangle $a = 6$ cm., $b = 10$ cm., $C = 48^\circ$.

Measure $\angle A$ of this triangle, measure also $\angle B$.

12. Consider again a triangle having $a = 6$ cm., and $b = 10$ cm., and lay down $CA = 10$ cm.

Where, then, does B lie?—It lies somewhere on a circle with centre C and radius 6 cm. (Fig. 12, drawn small).

By taking B in various positions, see how $\angle A$ varies, and again draw a graph giving $\angle A$ as a function of $\angle C$.

What is the position of B when $\angle A$ is greatest? And how would

you use Fig. 12 to find B for a given value of A ?

Use figures like Fig. 10 or Fig. 12 to make a graph giving $\angle B$ as a function of $\angle C$ (Fig. 11).

From Fig. 10 or Fig. 12, or from the graphs of A and B , make a graph of $A+B$ as a function of C ; and also a graph of $A+B+C$ as a function of C .

13. We discussed the making of a triangle with 2 sides and 1 angle given, the angle being opposite a given side, first by laying down the side opposite the given angle, and then by laying down the side next the given angle. Let us now try laying down the angle first. Take $a = 5.4$ cm., $b = 6.9$ cm.,

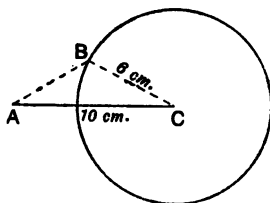


FIG. 12.

* The degree was introduced as a unit of angle to avoid fractions of a right angle. We now come to a fraction of a degree, which we may write decimally as above. Or we may now introduce another unit called a minute; 60 minutes make a degree, so that 37.5 degrees is 37 degrees 30 minutes, which may be written $37^\circ 30'$.

$B = 55^\circ$. Lay down the angle B equal to 55° . What is the next step?—Mark off BC equal to 5.4 cm. along one leg

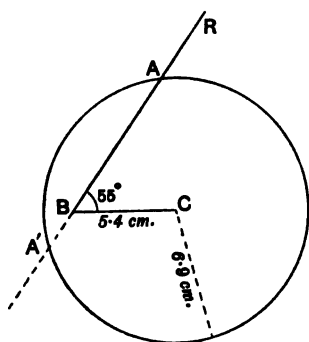


FIG. 13.

of the angle (Fig. 13, drawn small). Where must the point A lie?—It must lie on the other leg BR of the angle, and it must also lie 6.9 cm. from C , that is, on a circle with centre C and radius 6.9 cm. A must therefore lie at one of the intersections of the circle and the line.

Are the two intersections equally suitable?—The one marked A gives a triangle ABC having $a = 5.4$ cm., $b = 6.9$ cm., $B = 55^\circ$, so

that this intersection is suitable. The point marked A' gives a triangle $A'BC$ having the two sides of the right length, but the angle is not 55° but $180^\circ - 55^\circ$, or 125° , so that this intersection will not do, and we can make only one kind of triangle with the given values.

If you had another triangle DEF in which $EF = 5.4$ cm., $FD = 6.9$ cm., $E = 55^\circ$, could you fit it on to the triangle you have drawn?—Place F on C and E on B , which will make ED lie along BR . Then D must lie on the line BR and also on a circle with centre C and radius DF , that is, 6.9 cm. So D lies at one of the intersections of this line and circle; and A' is unsuitable, for this would make the angle $DEF = 125^\circ$ and not 55° ; so that D lies at A and the triangles fit.

14. Would these results be the same when a b B are given, no matter what values they have? Take $a = 5.4$ cm. and $B = 55^\circ$, and give b different values. Are the results the same as before when b is 14 cm.?—Yes; as before, A' is unsuitable, and there is only one triangle. Can you give any other values of b for which the results are the same as before?—Any value greater than 5.4 cm., that is, greater than a .

15. Are the results the same when $b = 4.8$ cm.?—

Having laid down the side a and the angle B , we have, as before, two loci on which A must lie, a straight line and a circle (Fig. 14); but in this case the drawing shows that the two intersections of the line and the circle lie on the same side of B . In consequence we have two triangles of different shapes, both of which satisfy the conditions; in the triangle ABC $BC = 5.4$ cm., $CA = 4.8$ cm., $\angle ABC = 55^\circ$; and in the triangle $A'BC$ $BC = 5.4$ cm., $CA' = 4.8$ cm., $\angle A'BC = 55^\circ$.

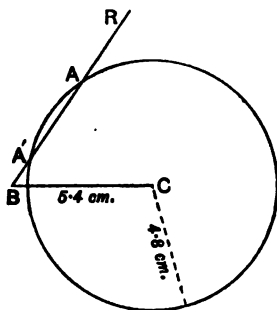


FIG. 14.

If you had another triangle DEF in which $EF = 5.4$ cm., $FD = 4.8$ cm., $E = 55^\circ$, could you fit it on to the triangle ABC ?—We fit E to B and F to C , and find that D must be at one of the intersections of the circle and straight line, namely, at A or A' ; but we have no means of knowing whether it will fall at A or A' . So that the triangle DEF will fit on to one of the triangles ABC and $A'BC$, but we cannot tell which.

Angles like $BA'C$ that are greater than 90° are called obtuse angles; angles like the other angles of the triangle $A'BC$, and like all the angles of the triangle ABC , that are less than 90° , are called acute angles. If we knew whether the triangle DEF has any angle obtuse, we could tell on to which of the triangles ABC and $A'BC$ it could be fitted.

16. Can you give any other values of b for which the results are the same as when $b = 4.8$ cm.?—As long as b is less than 5.4 cm., but great enough to make the circle with centre C and radius b cut the line BR , the results are the same.

As b gradually diminishes, what happens to the points A and A' where the circle cuts the line BR ?—They come closer and closer together till there is only one point, and then if b becomes still less the circle no longer meets the line.

Find by trial the value of b for which A and A' run together into one point.—It is about 4.4 cm.

Draw the circle with this radius 4.4 cm. When a straight line and a circle meet in only one

point, as in this case, they are said to touch one another, and the line is called a tangent to the circle. The point where they touch is called the point of contact.

17. Draw the triangle ABC (having $a = 5.4$ and $B = 55^\circ$) for the case in which A and A' run together so that there is only one point A ; and measure the angle BAC . Draw other figures with other values for a , and measure the angle BAC of each.

Consider again the case of Fig. 14. If we draw a perpendicular CN from C on BR , how does N lie with regard to A and A' ?—It lies midway between, since we know (chap. I, art. 14) that the perpendicular from the centre of a circle on a chord bisects the chord. Verify this by measurement.

If you draw circles with radii $5.2, 5.0, 4.6, 4.48, 4.44, \dots$ cm., all with C for centre, how would N lie with regard to the two intersections of any circle and the line?—It lies midway between.

And if a circle with centre C gradually diminishes in size till it only touches BR , how do its intersections with BR lie with regard to N ?—Always at the same distance on opposite sides, so that when they run together they must run together at N .

So that a tangent to a circle is perpendicular to the radius to the point of contact.

Ex. 2. Draw a number of angles of 55° . From any point C on one leg BC of each draw a perpendicular CN to the other leg BN . Measure CN and CB on each figure, calculate $\frac{CN}{CB}$ for each, and tabulate your results.

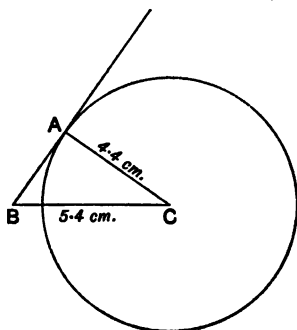


FIG. 15.

18. Draw a triangle ABC in which $a = 5.4$ cm., $b = 4.4$ cm., $B = 55^\circ$ (Fig. 15, drawn small); so that the construction gives a line and circle meeting in only one point A . If you made another triangle DEF with the same values of a b B , could it be fitted on to this triangle? Suppose $\angle E = 55^\circ$, $EF = 5.4$ cm., $FD = 4.4$ cm.—We lay EF along BC and

ED along BA . Then D must lie on the line BA and also on the circle with centre C and radius 4.4 cm. This line and circle have only one point common, namely A ; therefore D lies at A , and the triangles fit.

19. Try to draw a triangle having $a = 5.4$ cm., $b = 4.0$ cm., $B = 55^\circ$. What is the result?—We lay down $CB = 5.4$ cm., make $\angle CBR = 55^\circ$, and draw a circle with centre C and radius 4.0 cm. But this circle does not meet BR ; in fact we have seen that 4.4 cm. is the least radius that will make the circle cut BR . What, then, of the triangle you were asked to draw?—No triangle with the given values of a , b , B can be made.

Summarize the results that have been reached as to triangles that have $a = 5.4$ cm., $B = 55^\circ$, and b of some given length.—If b is greater than 5.4 cm. only one shape of triangle can be made; if b lies between 5.4 and 4.4 cm., triangles of two different shapes can be made; if b is less than 4.4 cm., no triangle can be made with the given values.

20. Discuss in the same way what kinds of triangles can be made with different values of b when

1. $a = 5.6$ cm., $B = 63^\circ$.
2. $a = 8.0$ cm., $B = 63^\circ$.
3. $a = 12.1$ cm., $B = 63^\circ$.
4. $a = 13$ cm., $B = 18^\circ$.
5. $a = 11.8$ cm., $B = 116^\circ$.
6. $a = 8.7$ cm., $B = 145^\circ$.

In cases 1, 2, 3, 4, the classification is the same as for the case discussed, in which a was $= 5.4$ cm., and $B = 55^\circ$.

In case 5 we lay down $CB = 11.8$ cm. and $\angle CBR = 116^\circ$ (see the sketch in Fig. 16). Then when b is greater than 11.8 cm. (say $= 17$ cm.) the circle with centre C and radius b meets BR in one point A , and meets BR produced back beyond B in a point A' . The triangle ABC has a , b , B of the required values; but the triangle $A'BC$ has the angle

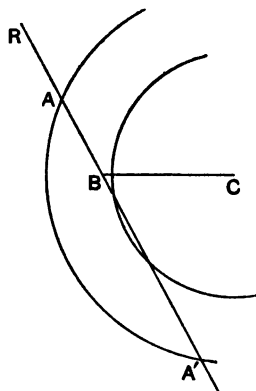


FIG. 16.

at B equal to 62° instead of 116° ; so that only one triangle can be drawn with the given values. When b is less than 11.8 cm. (say $= 11$ cm.) the circle with centre C and radius b does not meet BR at all, but only BR produced; so that the triangles it would give have the angle at B equal to 62° instead of 116° , and no triangle can be drawn with the given values.

Case 6 is similar to case 5.

If you have two triangles in which $a = 11.8$ cm., $b = 17$ cm., $B = 116^\circ$, can they be fitted together so as to coincide?—In the same way as before it is seen that they can.

21. Wherein do the data of cases 5 and 6 on the one hand and cases 1, 2, 3, 4 on the other differ, to account for these different results?—In cases 5 and 6 the angle B is obtuse, that is, greater than a right angle, while in the others it is acute, or less than a right angle.

If instead of $a b B$ you were given $b c C$ or $b c B$ or $a c A$, how far would the results differ?—To label the two sides and an angle opposite to one of them with the letters $a b B$ is only a convenient shorthand, and any other way of giving two sides and an angle opposite one of them amounts to the same thing.

Ex. 3. Take $B = 40^\circ$ and a in succession equal to 6, 9, 12, 15, 18 cm. In each case find the least value of b for which a triangle can be drawn. In each case calculate b as a fraction of a , and compare your results.

Repeat this, taking $B = 30^\circ$ instead of equal to 40° .

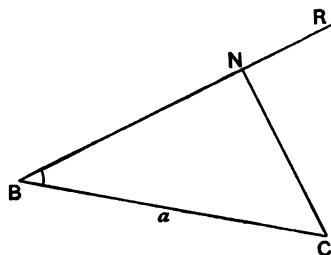


FIG. 17.

B is acute, and b is greater than a , there is one triangle

21. The results we have reached as to making a triangle when two sides and an angle opposite one of them are given can now be summarized. We continue to call these sides and angle $a b B$. Lay down the side CB or a and the angle CBB or B , and from C let fall CN perpendicular to BR (Fig. 17). When the angle

satisfying the conditions; when b is less than a but greater than CN , there are two triangles; when b is less than CN there is no triangle. When the angle B is obtuse, and b is greater than a , there is one triangle, but when b is less than a there is no triangle.

The results may be tabulated in the following form, the symbol ∞ denoting as great a number as you can imagine:—

Value of B	b lying between	Number of triangles with the given values
acute	0 and CN	0
	CN „ a	2
	a „ ∞	1
obtuse	0 and a	0
	a „ ∞	1

22. We have discussed the making of a triangle from 3 given sides, and from 2 given sides and 1 given angle. Next suppose only 1 side given and consider whether we can make up the data with angles. In copying the map (Fig. 1) we had such a case. What was it?—We made the triangle KWE by means of the length KW and the angles at K and W .

Then you can, out of a, b, c, A, B, C , give such a set of data to make the triangle ABC .—The set a, B, C would do.

Make a triangle having $a = 6.9$ cm., $B = 82^\circ$, $C = 60^\circ$. If you had a second triangle DEF in which $EF = 6.9$ cm., $E = 82^\circ$, $F = 60^\circ$, can you fit the two triangles together to coincide?—Place EF on BC . Then the angle E fits the angle B , and F fits C ; so that ED lies along BA and FD along CA , and D lies where BA and CA meet, that is at A ; so that the two triangles coincide.

Taking $a = 6.9$ cm. and $B = 82^\circ$, find by experiment for what values of C a triangle is possible.— C must be less than a certain angle which is about 98° .

When $a = 6.9$, $C = 60^\circ$, for what values of B is a triangle possible?— B must be less than 120° or so.

Experiment with a number of other cases and try to state the condition in a general form.—The sum of B and C must be less than 180° or so.

23. We have treated the case when the given side lies

between the given angles. Take a different case, and try to make a triangle having $a = 5.5$ cm., $A = 76^\circ$, $C = 50^\circ$.— Draw BC equal to 5.5 cm., and make an angle BCK of 50°

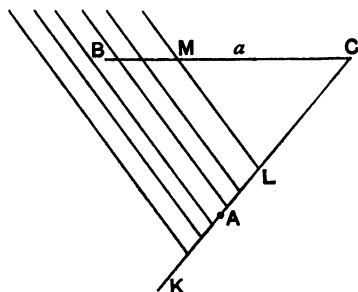


Fig. 18.

(Fig. 18). Now take any point L on CK and make an angle CLM equal to 76° . Take L in a number of positions and for each make $\angle CLM = 76^\circ$. Continue to try in this way until the line drawn at 76° passes through B . Denote the point L for this position by A , and we have the triangle ABC with the given values of a , A , C .

Draw another figure of BCK with $BC = 5.5$ cm. and $\angle C = 50^\circ$. On a piece of tracing-paper make an angle of 76° , lay it on your figure so that one leg points towards C from some point in CK , and move it about till the other leg passes through B . This gives another way of making the triangle.

Measure the angle B of the triangle. Measure also the angles which the line LM in its various positions makes with BC .

24. Taking $a = 5.5$ cm., $C = 50^\circ$, find experimentally for what values of A a triangle is possible.— A must be less than about 130° . Experiment with other values of a and C , and try to reach a general statement of the condition.— $A + C$ must be less than about 180° .

If you have two triangles ABC and DEF in which $BC = EF = 5.5$ cm., $\angle C = \angle F = 50^\circ$, $\angle A = \angle D = 76^\circ$, can they be fitted together to coincide?—Suppose that in making ABC we used the angle of 76° drawn on tracing-paper, and made the point L (Fig. 18) slide along from C to K , the first leg of the angle lying all the time along LC . Then the point M where the second leg cuts BC travels all the time from C towards B and then beyond B . The point A is the position occupied by L at the moment when M reaches B , and since L travels all the time away from C there is only one such

point. Now lay EF on BC and the triangle DEF on ABC so that the angles F and C fit together. If now the tracing-paper with the angle of 76° is moved as before, the position of D is found just as A was, and is at the same point.

25. We have tried to make a triangle from 3 sides, from 2 sides and an angle, and from 1 side and 2 angles. Can it be made from 3 angles? Try with the values $A = 20^\circ$, $B = 30^\circ$, $C = 50^\circ$. Make an angle of 20° and call the legs AK and AL (Fig. 19). No length is given for AB , so try

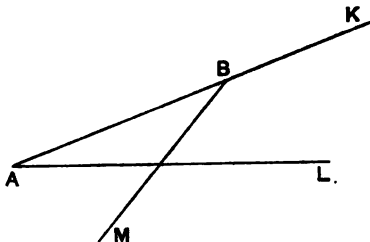


FIG. 19.

taking B at random on AK , and make the angle $ABM = 30^\circ$. How will you complete the triangle?— C must be where AL and BM cross, so that the triangle is already completed. Is the angle C equal to 50° , as it was to be?—No, it is about 130° .

See if you get any better result by taking B at other positions on AK , or by beginning with another angle than A .

Try making triangles with other values for the angles:—

$$A = 50^\circ, \quad B = 70^\circ, \quad C = 60^\circ;$$

$$A = 50^\circ, \quad B = 70^\circ, \quad C = 80^\circ;$$

$$A = 20^\circ, \quad B = 50^\circ, \quad C = 110^\circ.$$

What is the result?—Sometimes the third angle has the given value, sometimes not, but the third angle is always settled when the first two are made.

Can you (without drawing) predict what C will be when $A = 30^\circ$ and $B = 40^\circ$? Experiment with different values of the angles until you can predict the third angle from the given values of two.

26. You made several triangles having $A = 20^\circ$ and $B = 30^\circ$. Call the sides of one of them a b c , and of

another $a' b' c'$. Measure all the six sides, calculate as decimals the three quantities $\frac{a}{a'}, \frac{b}{b'}, \frac{c}{c'}$, and compare the three results. What do you notice?

Repeat with a different pair of the triangles that you have drawn.

Repeat again with two triangles, both of which have $A = 48^\circ$ and $B = 57^\circ$, or any other values.

Draw any triangle and measure its sides $a b c$. Make another triangle with sides equal to $1.4a, 1.4b, 1.4c$. Measure the angles of the two triangles and compare them.

Repeat, using a different number in place of 1.4.

These results will be discussed later.

27. Let us summarize our results. Of the 6 elements of a triangle—the 3 sides and 3 angles—any 3 that include at least one side are enough to know in order to make the triangle. In all cases but one the triangle is unique; that is, with the given values only one shape of triangle is possible. The one exception is that in which 2 sides and the angle opposite one of them are given, the angle is acute, and the side opposite this angle is less than the other given side; in this case two shapes are possible.

When two triangles are such that one could be placed on the other so as to fit exactly, the two triangles are called congruent triangles.

28. *The Slider-Crank Pair.*—In many machines there occurs an arrangement of three pieces known as slider, crank, and connecting rod. In Figure 20, the slider C may

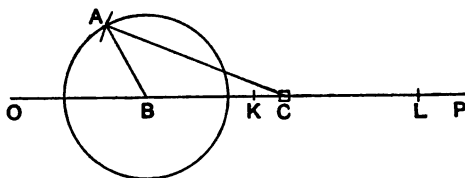


FIG. 20.

slide to and fro along the line OP , the crank BA is a rod that turns about a fixed point B in the line OP , and the

connecting rod AC is a rod connecting the slider C to the end A of the crank.

Take the crank 7 cm. long, and the connecting rod 16 cm. long. What more must you know to fix the position of the system? Give the additional condition in as many different ways as you can.

Show what part KL of the line OP the slider C is able to traverse. Find the position of the system (1) when C is 12 cm. from B , (2) when C is 6 cm. from its end position K , (3) when the crank makes 40° with BP , (4) when crank and connecting rod are at right angles, (5) when the connecting rod makes 20° with CB . In each case see how many solutions there are, that is, how many positions satisfy the conditions.

By the help of measurements of the figures you have drawn and of other figures, make graphs giving the angles A B C of the triangle ABC as functions of the distance KC . Also make a graph of $A + B + C$ as a function of the distance KC .

EXERCISES.

4. If you were told there were 3 towns A B C such that from A to B was 5 miles, B to C was 6 miles, and C to A was 12 miles, what would you say?

If you were told that the distance BC was 6 miles, CA 12 miles, and the angle BCA 25 degrees, what would be the distance AB ?

5. Draw triangles all having $a = 6$ cm., and $b = 8$ cm., while the angle C has in different triangles various values between 0° and 180° , and measure c in each.

Draw a graph giving c as a function of C .

Calculate the area of a square on the side c of each triangle you have drawn, and show this area as a function of C . On the same graph show the values of the squares on a and b , namely 36 and 64 sq. cm.

6. (1) Construct a triangle PQR , with base $PQ = 9$ cm. and $\angle RPQ = 42^\circ$.

(2) On the same base construct a triangle PQS with $PS = 7$ cm.

Draw the locus of R in (1) and of S in (2).

On your drawing mark the vertex T of a triangle PQT , which has $\angle TPQ = 42^\circ$ and $PT = 7$ cm.

State a set of data sufficient to determine both the shape and size of a triangle; and explain why the 3 angles are insufficient.

7. A square reservoir has been made in a field and is to be covered up. It is, therefore, necessary to take measurements from which the positions of its corners may be found again. Supposing the field rectangular, give the least number of measurements (in addition to the known length of a side of the reservoir) that must be made. Give two sets of measurements either of which would do.

8. A triangle ABC is supposed to be cut out of paper and laid down on a sheet of paper, the corners $A B C$ being marked on the triangle. The triangle is first turned over about the side BC ; it is next turned over about the side CA , and finally it is turned over about the side AB . Draw any triangle for the first position, and then draw the triangle in its final position.

9. Construct the quadrilateral $ABCD$ having $AB = 5.3$ cm., $BC = 7.2$, $CD = 9.0$, $DA = 4.8$, $\angle ABC = 90^\circ$. Find by measurement the distance of D from the diagonal AC and determine the area of the figure.

Now suppose AB and BC to be fixed, CD to revolve about C , and AD to be of variable length. Show the position of CD when the area of $ABCD$ is a maximum, and give the maximum area.

10. Figure 21 shows a piece of ground, divided by hedges into

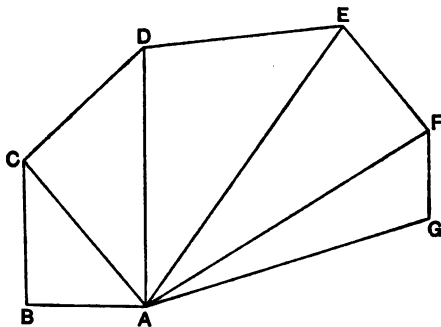


FIG. 21.

triangles. How many measurements would have to be made in order to draw a plan of the triangle ABC ? And how many for the figure $ABCD$? And how many for the figure $ABCDE$? In each case give one set of measurements (not more than are needed) and tell how you would use them to draw the plan.

If the figure was made up in the same way of n triangles, how many measurements would have to be made? And how many sides would the outer boundary of the piece of ground have?

How many measurements would be necessary for a piece of ground whose boundary had m sides?

11. In Figure 22 ABC and PQR are triangles in which $AB = PQ$, $BC = QR$, $CA = PR$. Prick the figure off on a sheet of paper, and

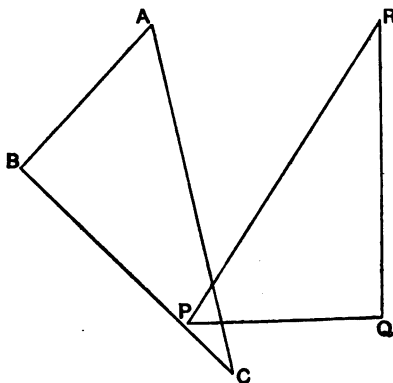


FIG. 22.

draw the locus of points equidistant from A and P , and the locus of points equidistant from B and Q . Show that the point where these loci meet is equidistant from C and R .

12. A stone is thrown horizontally from the top of a vertical cliff 50 metres high with a speed of 11 metres per second. Its horizontal distance from the top of the cliff increases steadily so that after t seconds the distance is $11t$ metres. Its vertical depth below the top of the cliff increases in such a way that after t seconds the depth is $4.9tt$ metres. Calculate the position of the stone at various times, and show its path on a scale of 1 to 300, that is, of 1 cm. to 3 metres.

13. The distances and second class railway fares from Paris to the under-mentioned stations are as follows, the distances being reckoned in miles and the fares in francs and centimes (1 franc = 100 centimes) :—

	Distance in		Fare.	
	Miles.		fr.	c.
St. Denis	4		0	55
Chantilly	$25\frac{1}{2}$		3	10
Creil	32		3	85
Longueil	45		5	45

	Distance in		Fare.	
	Miles.	fr.	c.	
Compiègne	53	6	35	
Noyon	67	8	15	
Chauny	77	9	35	
Tergnier	82	9	90	
St. Quentin	96	11	65	

From these data draw on squared paper a diagram showing the relation between the distance travelled from Paris and the railway fare. Examine whether the railway fare is proportional to the distance travelled, assuming distances to be given to the nearest half mile, and 5 centimes to be the smallest coin used for payment of railway fares.

N.B.—A diagram is meaningless unless the scale of measurement is stated and the positions of the lines from which distances are measured are indicated.

14. In a field $ABCD$ AB is 5·63 chains, AC 10·2 chains, AD 10·9 chains, the angle BAC $27^{\circ} 15'$, and the angle CAD 33° . Draw a diagram to a scale of 2 chains to the inch, and find the area to the nearest rood.

15. Find the discharge per minute in gallons from a full cylindrical pipe of 6 inches diameter, the mean velocity being 12 inches per second.

CHAPTER VI

FURTHER PROBLEMS IN MENSURATION

1. A GALLON is 277 cubic inches, and the standard gallon measure is a vessel with a circular bottom and vertical sides, the height of the vessel being equal to the diameter of the bottom. Suppose we want to make a gallon measure of the standard shape, and let us consider how we may determine its height and diameter from these data.

What do you know of the capacity of a cylindrical vessel?—That the capacity is the product of the height h by the area A of the base. And when the base is circular what do you know of its area?—That it is $8\cdot14 \times r \times r$ if r is the radius, or $0\cdot785 \times d \times d$ if d is the diameter.

What then is the capacity of a cylindrical vessel h inches high and d inches in diameter?—It is $0\cdot785 \times d \times d \times h$ cubic inches.

And the capacity of a cylindrical vessel whose height and diameter both measure d inches?—It is $0\cdot785 \times d \times d \times d$ cubic inches, or more shortly written $0\cdot785 ddd$ or $\frac{1}{4}\pi ddd$.

Now how will you find out what d must be to make this capacity 277 cubic inches?—We could try various values for d and calculate the corresponding values of $0\cdot785 \times d \times d \times d$.

Do so for a number of integers 1, 2, 3, . . . , and find the two values of $0\cdot785 ddd$ between which 277 lies. They are given by $d = 7$ and $d = 8$. You could now calculate the capacity for d equal to 7·1, 7·2, . . . 7·9, and thus get still nearer to the capacity 277 c. in. ; but few pupils are fond enough of arithmetic to make this worth doing.

2. Could you use a graph to find the value of d that will make $0\cdot785 ddd$ equal to 277?—Yes, we could draw the graph of $0\cdot785 ddd$ and read off the value of d for which it is equal to 277.

As it takes some time to draw a graph, perhaps you could use one you have already drawn ; look through them and see if any of them will help.—There is a

84 FURTHER PROBLEMS IN MENSURATION

graph giving the volume of a cube for various values of the edge of the cube, that is, giving ddd for various values of d ; and the value of d that makes $ddd = 277/0.785$ will make $0.785 ddd = 277$; so that we simply read off the value of d that makes $ddd = 353$.

Such a relation as $0.785 ddd = 277$ in which two things are stated to be equal is called an equation; a value of d which makes the two things equal is called a root of the equation; and the process of finding such a value of d is called solving the equation.

3. Find the height and diameter of a standard litre measure, that is, of a cylindrical vessel that holds 1000 cubic centimetres and whose height is twice its diameter.

Thermometers.

4. Look at a Fahrenheit thermometer. It tells us how warm the room is. The mercury (or other liquid) inside the tube stands opposite a mark on the scale. These marks all have numbers, some of which are shown; there is not room to show all of them. If it stands opposite 55 we say the temperature is 55 degrees, or more completely 55 degrees on the Fahrenheit thermometer, which is written shortly 55°F . If in the morning the mercury stands lower, for instance at 40° , it is too cold, and the room is artificially heated till the reading of the thermometer is about 55, that is, till the temperature is about 55° . If the temperature is 70° before the room is heated, it is already warmer than we need, and we do without further heating.

A man called Fahrenheit invented this thermometer and called the lowest temperature he could reach by mixing snow and salt 0° , while he called the temperature of his body 100° . He marked with 0 the point at which the liquid stood when the thermometer was plunged in the mixture of snow and salt, and with 100 the point at which it stood when at the temperature of his body. The interval between these he divided into 100 equal parts and marked with numbers from 0 to 100. The graduation of the thermometer was carried beyond 100 by marking off further divisions equal to those from 0 to 100.

It was found that water froze at the temperature called 32

on this thermometer and boiled at 212. The freezing and boiling points are more definite than the temperature of the snow-and-salt mixture and the temperature of the body, and a Fahrenheit thermometer is now made by fixing these freezing and boiling points, calling them 32 and 212, dividing the interval between into 180 parts, and continuing these graduations at both ends of the interval. It thus happens that on the Fahrenheit thermometer the temperature of the body is not exactly 100 but somewhat lower.

5. Look now at a Centigrade thermometer. On it the freezing point of water is called 0° and the boiling point 100° . How many degrees was it on the Fahrenheit thermometer from the freezing to the boiling point?—It was 180 degrees. And how many Fahrenheit degrees are equal to a Centigrade degree?— $\frac{180}{100}$ or 1.8 or $\frac{9}{5}$ Fahrenheit degrees.

Blood heat or the temperature of the body is 37 degrees on the Centigrade scale, or more shortly is 37°C . On the Fahrenheit scale how many degrees above freezing point is this temperature which is 37 degrees above on the Centigrade scale?—It is 37×1.8 or 66.6 degrees Fahrenheit above freezing point. The Fahrenheit scale beginning 32 degrees below freezing point, what is this temperature called on that scale?—It is called $66.6 + 32$ or 98.6 degrees.

If a temperature is $x^{\circ}\text{C}$., that is, is x degrees above freezing point on the Centigrade scale, how many degrees is it above freezing point on the Fahrenheit scale?—It is $x \times 1.8$ degrees above freezing point on the Fahrenheit scale. And what is the temperature on the Fahrenheit scale, that is, when the mercury stands at the point marked x on the Centigrade scale, what is the mark y at which it stands on the Fahrenheit scale?—It is $x \times 1.8 + 32$. So that in mathematical shorthand we write

$$y = x \times 1.8 + 32.$$

6. Sea water boils at a temperature of 103.7°C ., turpentine at a temperature of 156°C ., mercury at a temperature of 350°C .; lead melts at a temperature of 325°C ., and zinc at a temperature of 391°C . Give these temperatures on the Fahrenheit scale.

86 FURTHER PROBLEMS IN MENSURATION

7. The melting point of tin is 442°F . How will you find the temperature Centigrade at which it melts?—The temperature is $442 - 32$ or 410 Fahrenheit degrees above the freezing point of water. And each of these degrees is $\frac{1}{1.8}$ of a degree Centigrade, so that it is $410 \div 1.8$ or 228 degrees Centigrade above the freezing point, that is, the temperature is 228°C .

Could you have found this result from the relation

$$y = 1.8x + 32$$

by putting 442 for y and then trying to find a value of x that will make the relation true? That is, can you find a value of x that will make $442 = 1.8x + 32$?—The value of x that does this will make $442 - 32$ or 410 equal to $1.8x$, or will make $410 \div 1.8$ or 228 equal to x . Wherein does this differ from the previous method of finding the melting point of tin on the Centigrade scale?—It differs only in treating all the quantities as mere numbers, without reference to their physical meaning.

8. The process of finding a value of x that will make $442 = 1.8x + 32$ is called solving the equation

$$442 = 1.8x + 32,$$

and the value 228 is called the root of the equation. Can you solve the equation $y = 1.8x + 32$ as an equation in x , that is, can you put the relation in such a form as to show at once the operations to be performed to give x as soon as we know what y is? In other words, can you express the Centigrade temperature x in terms of the Fahrenheit temperature y ?—The relation $y = 1.8x + 32$ will be true if $y - 32$ is equal to $1.8x$. And these are equal if $(y - 32) \div 1.8$ is equal to x . The relation is now in the required form $x = (y - 32) \div 1.8$, which directs us to subtract 32 from y and then divide by 1.8 .

9. Reason out this expression for the Centigrade temperature in terms of the Fahrenheit temperature from the way the thermometers are made and without reference to equations.

10. On both thermometers low temperatures are shown by continuing the uniform divisions of the scales below the

mark 0, and numbering them 0, -1, -2, -3, and so on. What is the temperature Centigrade of the mixture of snow and salt which is marked 0 on the Fahrenheit scale?—It is 32 degrees Fahrenheit below the freezing point. And 32 degrees Fahrenheit are $32 \div 1.8$ or 17.8 degrees Centigrade, so that it is 17.8 degrees Centigrade below 0°C ., that is, the temperature is -17.8°C .

11. Could you have used to find this temperature either of the relations $y = 1.8x + 32$ and $x = (y - 32) \div 1.8$, which were found for positive readings, that is, for temperatures above 0 on the two scales?—In the present case y is 0, so that the relations are $0 = 1.8x + 32$ and $x = (-32) \div 1.8$.

Now we have not hitherto met with any operation requiring the division of a quantity such as -32, and it has at present no meaning. Can you assign it a meaning that will give x its proper value -17.8 ?—If we suppose $(-32) \div 1.8$ to have the same meaning as $-(32 \div 1.8)$, that is, if we take $(-32) \div 1.8$ to mean that 32 is to be divided by 1.8 and then to have the minus sign — put before it, this supposition will give x its proper value $-(32 \div 1.8)$ or -17.8 .

Again look at the thermometers. We may take -32 to mean that from the freezing point we must, on the Fahrenheit scale, go 32 degrees down to get to the point in question, and $(-32) \div 1.8$ to mean that we must go $32 \div 1.8$ degrees down on the Centigrade scale. Which agrees with making $(-32) \div 1.8$ mean the same as $-(32 \div 1.8)$.

12. On the Centigrade scale the freezing point of mercury is -39.5° . Find it on the Fahrenheit scale.—It is 39.5 degrees Centigrade below the freezing point of water; which is 39.5×1.8 or 71.1 degrees Fahrenheit below. And the Fahrenheit zero is 32 degrees below, so that the point required is $71.1 - 32$ or 39.1 degrees below the Fahrenheit zero. That is, the freezing point of mercury is -39.1°F .

Try whether the relation $y = 1.8x + 32$ gives this result.—Here $x = -39.5$, so that $y = 1.8 \times (-39.5) + 32$. Here again you have a new form $1.8 \times (-39.5)$. Let us take -39.5 as a direction to go 39.5 degrees down the Centigrade scale from the freezing point. This means going 39.5×1.8 or 71.1 degrees down the Fahrenheit scale, so we may appropriately

88 FURTHER PROBLEMS IN MENSURATION

make $1.8 \times (-39.5)$ mean $-(1.8 \times 39.5)$ or -71.1 . Then $+32$ means go 32 degrees up again, so that we arrive at the point $71.1-32$ or 39.1 degrees below the freezing point, or at the point called $-71.1+32$ or -39.1 .

13. If the same temperature is p degrees below zero on the Fahrenheit scale and q degrees below zero on the Centigrade scale, find the relation between p and q , giving it in the two forms, one suited for the calculation of p when q is known and the other for the calculation of q when p is known. As a particular case consider the freezing point of mercury, for which $p = 39.1$.

14. The freezing point of mercury has $p = 39.1$ and $q = 39.5$, so that for this case p and q are nearly equal. What temperature has p and q exactly equal? Use the relation you have found, namely, $q = (p + 32) \div 1.8$ or $p = 1.8q - 32$.—Let us suppose p and q to belong to the temperature we are in search of. We separate the term $1.8 \times q$ into two parts $1.0 \times q$ and $0.8 \times q$. The former is equal to p ; leaving $0.8 \times q$ to be made equal to 32, in order to make $p = 1.8q - 32$. And $0.8 \times q$ will be $= 32$ if q is $= 32 \div 0.8$ or 40. Reasoning back again we have $0.8 \times 40 - 32 = 0$ and $40 + 0.8 \times 40 - 32 = 40$, that is, $1.8 \times 40 - 32 = 40$. So that -40° F. and -40° C. mean the same temperature.

Or we could proceed more formally to solve the equation $1.8q - 32 = p$ when $p = q$. We have to find a value of p that will make $1.8p - 32 = p$. The two quantities $1.8p - 32$ and p being equal will still be equal if we subtract p from each of them. They then are $0.8p - 32$ and 0, so that we want to make $0.8p - 32$ equal to 0, or to make $0.8p$ equal to 32, and these last are equal if $p = 32 \div 0.8$ or 40.

15. We had the relation $y = 1.8x + 32$ between the Fahrenheit reading y and the corresponding Centigrade reading x . When these readings x and y are the same what is their value? That is, what value of x makes $x = 1.8x + 32$?

Let us subtract x from each of the two equal quantities x and $1.8x + 32$. They became 0 and $0.8x + 32$. Now $0.8x + 32$ is positive for all positive values of x and even

when x is zero, so that no positive value of x can make $0.8x + 32$ equal to 0. But let us allow negative values and treat them as before. Since $0.8 \times 40 = 32$ we have $0.8 \times (-40)$ equal by our agreement to $-(0.8 \times 40)$ or -32 . So that $x = -40$ makes $0.8x + 32 = 40$, and makes

$$x = 1.8x + 32.$$

16. *Negative numbers.*—A boy has 20 shillings and spends 15. How many has he left?— $20 - 15$ or 5. And if he now buys 7 shillings' worth of things?—His problem is to pay 7 shillings out of 5, or to find the value of $5 - 7$; he has to subtract 2 from 0. He is 2 shillings in debt. If now some one gives him 10 shillings?—He pays his debt and has 8 shillings left.

In the case of the thermometer, distances above the zero were marked with ordinary or positive numbers, distances below with negative numbers. Let us in the present case, instead of saying the boy is 2 shillings in debt, say 'he has -2 shillings'. Then we may say the number of shillings he had

at first was	20,
after the first transaction	$20 - 15$ or 5,
" " second "	$20 - 15 - 7$ or -2 ,
" " third "	$20 - 15 - 7 + 10$ or 8.

Suppose the boy began with 20 shillings, and spent a shillings, then b shillings, and then was given c shillings. How many had he after each transaction?—The numbers of shillings he has at the various stages are

$$\begin{aligned} &20, \\ &20 - a, \\ &20 - a - b, \\ &20 - a - b + c. \end{aligned}$$

As long as we confined ourselves to ordinary numbers any of these expressions that were not positive were meaningless, but as soon as we make negative numbers mean debt these expressions have meaning whatever the values of a and b . Thus if a is 25, so that $20 - a = -5$, we find that after the first transaction the boy 'has -5 shillings', which we understand at once to mean he 'is 5 shillings in debt'.

Sheep enclosure.

17. A farmer has sheep hurdles of a total length of 100 yards, which he sets up to enclose a rectangular piece of ground. How will the area enclosed vary as the shape of the enclosure varies? And when will the area be greatest?

If one side is 10 yards long, what are the lengths of the other three, and what is the area?—The sides are 10, 40, 40 yards, and the area 400 square yards.

Calculate the area, supposing in succession that one side has the lengths 10, 15, 20, . . . 40 yards, and draw a graph showing the area as a function of the length of this side. From the graph read off the length for which the area is greatest; what are then the lengths of the other three sides?

If one side is x yards long what are the other sides, and the area?—The other sides are x , $50-x$, $50-x$ yards and the area $x \times (50-x)$ square yards. Sketch the enclosure (Fig. 1), and from your sketch give another form

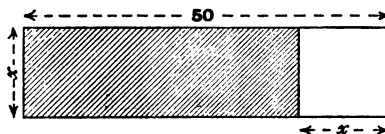


FIG. 1.

for the area.—It is $50 \times x - x \times x$ or $50x - xx$ square yards.

Can you draw the graph of $50x - xx$?—It is done already, it is the graph of the preceding paragraph.

Can you find a value of x that will make $50x - xx$ equal to 520?—From the graph we see that $x=35$ will approximately make $50x - xx = 520$.

By putting $x = 35.2$ and $x = 35.8$ show that the value of x lies between these two values. What problem connected with our sheep enclosure is this that we have solved?—We have found the shape of the enclosure when its area is 520 square yards.

What other value for x does the graph give for an area of 520 square yards, and how is it related to the former, and what is the shape of the enclosure for this value?

18. When the enclosure is square what is the length of each side?—25 yards. And if one side is y yards

more than 25, what is the length of a side next it?—It is $25 - y$ yards.

And the area?— $(25 + y) \times (25 - y)$ square yards.

Sketch the enclosure (Fig. 2), and give the area in another form.—

The area is $25 \times 25 - y \times y$ square yards or $625 - yy$ square yards.

Show in a graph the area as a function of y . For what value of y is the area greatest? For what value of y is the area 520 square yards?

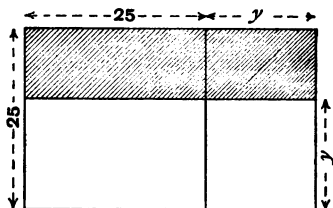


FIG. 2.

Some time ago we had a graph for the area of a square in terms of the length of the side. Could you use this to find what value of y makes $625 - yy = 520$?—Yes, it is the same value that makes $yy = 625 - 520$, that is, $= 105$, and from the graph we find this value to be $10\cdot2$.

19. On the same diagram draw the graphs of xx and $50x - 520$. Give the distances of a point where the two graphs cross from the two axes, that is, from the two lines from which all distances are measured. Use these graphs to solve the equation $50x - xx = 520$.

20. By means of any of the methods discussed, can you find how the hurdles must be arranged to give an area (1) of 100 square yards, (2) of 400 square yards, (8) of 700 square yards? One of these areas is impossible; in this case apply each method and show why it fails to give a result.

21. In discussing enclosures made with 100 yards of hurdles we used x for the length (in yards) of a side and y for the length by which a side exceeds 25. What is the relation between x and y , when they refer to the same enclosure?—It is $x = 25 + y$.

What other relation do you get between x and y by expressing the area of the enclosure in terms of x and of y ?—We get $50x - xx = 625 - yy$.

Suppose we have 200 yards of hurdles and use x and y in the same way. What is the meaning of y and what is the

92 FURTHER PROBLEMS IN MENSURATION

relation between x and y ?—A square made with these hurdles has a side of 50 yards, so that y is the excess of a side over 50; and $x = y + 50$. And what relation between

x and y do we get from the expressions for the area?—We get

$$100x - xx = 2500 - yy.$$

What are these relations if we have $4 \times a$ yards of hurdles?—They are $x = a + y$ and $2ax - xx = aa - yy$.

22. We have seen how to use the graph of yy to find a value of x that will make $50x - xx = 520$. Can you use the same graph to find a value of x that will make $8x - xx = 1$?

This will be our old problem if we make the area 1, and the distance round the rectangle 2×3 , in any corresponding units, for instance in square yards and yards or in square cm. and cm. Then y is taken so that $x = \frac{3}{2} + y$ or $y = x - 1.5$, and the area-relation is $8x - xx = 1.5 \times 1.5 - yy$. We have then to make $1.5 \times 1.5 - yy$ or $2.25 - yy$ equal to 1, or to make yy equal to 1.25. The graph of yy shows that yy has this value when $y = 1.12$. Then for x we have

$$x = y + 1.5 = 1.12 + 1.5 = 2.62.$$

Therefore 2.62 is a value of x that makes $8x - xx = 1$.

Verify this result by calculating the value of $8x - xx$ when $x = 2.62$.

23. If you try in the same way to find a value of x that makes $8x - xx = 4$ (that is, to solve the equation $8x - xx = 4$), what difficulty do you meet?

24. We have seen that approximately $50x - xx = 520$ when $x = 35$. Calculate the value of $100x - 2xx$ when $x = 35$. Can you give a value of x that will approximately make $100x - 2xx = 1040$?—We have it already, namely, 85.

Can you solve the equations:

$$150x - 3xx = 1560;$$

$$200x - 4xx = 2080;$$

$$500x - 10xx = 5200;$$

$$5x - xx \div 10 = 52;$$

that is, can you find for each a value of x that will make the relation true?

Again we saw that $3x - xx = 1$ when $x = 1.12$. Solve the equations:

$$\begin{aligned} 8x &= xx + 1; \\ xx + 1 - 8x &= 0; \\ 30x &= 10xx + 10; \\ 21x - 7xx &= 7; \\ 6xx &= 18x - 6. \end{aligned}$$

25. Take now the equation

$$40x - 8xx = 5$$

and try to solve it. How will you proceed?—We could draw the graph of $40x - 8xx$ and read off a value of x that makes it equal to 5. Or we could draw the graphs of $8xx$ and $40x - 5$ on the same diagram; then a value of x for which the two graphs cross gives a root, that is a value for which $8xx = 40x - 5$.

Could you solve the equation by means of the graph used before, that of yy ? Could you treat this equation as a problem about the shape of a rectangle whose perimeter (that is, the sum of the sides) and area are given? What is there about this equation that keeps us from treating it exactly as we treated $3x - xx = 1$?—The difference lies in having $8xx$ instead of xx .

Can you get rid of this 8?—We have three areas, $40x$, $8xx$, and 5, and the difference between $40x$ and $8xx$ is to be equal to 5. If we take a third of each area they will be related in the same way, that is, the difference between $\frac{40}{3}x$ and xx will be equal to $\frac{5}{3}$. So we may treat the equation $\frac{40}{3}x - xx = \frac{5}{3}$ instead of $40x - 8xx = 5$.

26. What, then, is the equation $\frac{40}{3}x - xx = \frac{5}{3}$ as a problem on a rectangle of given area and perimeter?—The problem is to find the dimensions of a rectangle of perimeter $\frac{40}{3}$ and of area $\frac{5}{3}$. In this case $x = \frac{20}{3} + y$ and

$$\frac{40}{3}x - xx = \frac{400}{9} - yy.$$

So that yy must be equal to $\frac{400}{9} - \frac{5}{3} = 42.8$, and the graph gives y equal to 6.5. This gives $x = 6.7 + 6.5 = 13.2$ for one side of the rectangle, while the other side is 0.1.

Find the value of $40x - 8xx - 5$ for $x = 13.2$ in order to see if 13.2 is really a root of the equation. Find

94 FURTHER PROBLEMS IN MENSURATION

the value also for $x = 18.21$ and so show that the exact value of x lies between 18.20 and 18.21.

27. Of the methods used above to solve equations some give two roots, that is, two values of x that make the relation true, some only one. In all cases the rectangle (when there is a rectangle at all) has two unequal sides. In each case where only one side of the rectangle is found as root, find the length of the other side and see whether it is also a root.

EXERCISES.

1. It is known that a jam tin a inches high and b inches in diameter will hold $0.037 \times a \times b \times b$ pounds of jam. Find whether a tin 2.8 inches high and 2.8 inches in diameter will hold a pound of jam.

2. I want to buy about 20 yards of certain iron wire, but I find it is sold only by the pound. The wire is 0.14 of an inch in diameter, and iron weighs 470 pounds a cubic foot. The area of the section of a piece of wire d inches in diameter is $0.785 \times d \times d$ square inches. How many pounds must I ask for?

3. The weight in pounds of a foot of iron piping is given by the formula $\frac{\pi tw}{144}(b+t)$ when the thickness is t inches, and the bore is b inches; w is the weight in pounds of a cubic foot of the iron. With $b = 3$, $t = \frac{3}{8}$, the weight of a foot is found to be 13.3 lb.; find w . Hence calculate, to the nearest pound, the weight of a foot if $b = 2\frac{1}{2}$, $t = \frac{1}{4}$.

4. If v and V are the volumes of a certain quantity of gas, under constant pressure at 0° and t° Centigrade, respectively,

$$V = v \times \left(1 + \frac{t}{273}\right).$$

Find, to the nearest tenth of a cubic centimetre, the volume occupied at 18° Centigrade by some gas which occupied 100 c.c. at 0° Centigrade.

5. If an oak beam L inches long, B inches wide and D inches deep is fixed at one end and a weight W cwt. is placed at the other end, it will break should W exceed

$$\frac{15 \times B \times D \times D}{L}.$$

Find with as little work as possible whether such a beam 10 feet 8 inches long, 1 foot 10 inches wide, and $4\frac{1}{2}$ inches deep will break if a weight of 2 tons is placed at the free end.

6. The effective horse-power of a certain kind of water-wheel is $0.0066 \times Q \times h$, where Q is the supply of water in cubic feet per min. and h is the head of water in feet.

Find the effective horse-power in a case where the head is 5 feet and the water is sufficient to make a stream 10 feet wide, 2 feet deep and flowing at the rate of 5 feet per second.

7. a horses are bought at £ x each, b horses at £ y each, and c at £ z each. What is the average cost per horse? If, when re-sold, the gain on each is one-tenth of its cost price, what is the total gain and the total selling price?

Give results, both for the general case, and for the special case in which $a = 7$, $b = 5$, $c = 8$, $x = 20$, $y = 36$, $z = 45$.

8. A steam plough ploughs an acre in a minutes. Express, in hours, the time it will take to plough a field b yards long and c yards wide.

9. If a man walk p miles in q hours, find how far he will walk in 5 hours; how far he will walk in x hours; how long he will take to walk 5 miles; how long he will take to walk y miles?

10. A steamer goes a miles up stream in b hours, and then c miles down in d hours. The stream flows e miles an hour and the boat could go f miles an hour in still water. Express a and c in terms of b, d, e, f .

If $a = 14$, $b = 2$, $c = 14$, $d = \frac{1}{2}$, what are e and f ?

11. In order to find the diameter of a tube of uniform bore some mercury was poured into it and the height of the column measured. The weight of the mercury was 25.6 grams, the height of the column 15.3 cm. What was the diameter of the tube? A cubic cm. of mercury weighs 13.6 grams.

12. A pint of water weighs 1.125 pounds, and a pint of milk between 1.155 and 1.165 pounds. A mixture of the two weighs 1.148 pounds per pint. Between what limits does the percentage of milk by volume lie?

13. A shot-gun, into whose barrel will exactly fit a sphere of lead weighing $\frac{1}{N}$ th of a pound, is termed

an ' N -bore' gun. The diameter of the barrel of a 12-bore gun is 0.729 inch. What is the diameter of the barrel of a 20-bore gun?

14. $ABDC$ is a square field (see rough sketch Fig. 3), each

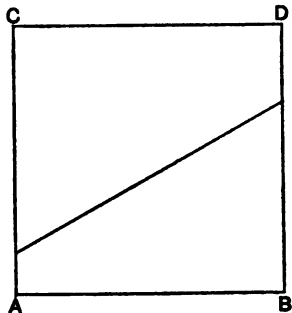


FIG. 3.

96 FURTHER PROBLEMS IN MENSURATION

side being 120 yds. AB runs east, and AC north. A footpath runs across the field, and one point of it is 20 yds. distant both from AB and from AC . If by following the footpath you find at any moment that the distance you have advanced north is $\frac{2}{3}$ the distance you have advanced east, show by a diagram drawn on squared paper to a scale of 20 yds. to an inch, the exact position of the footpath, and give the distance from B at which the footpath leaves the field.

If x yards is the distance of any point of the path from AC and y yards its distance from AB , what relation between x and y will hold true for all points of the path?

15. Find in sq. feet the area of the kite shown in Fig. 4 on a scale of an inch to a foot.

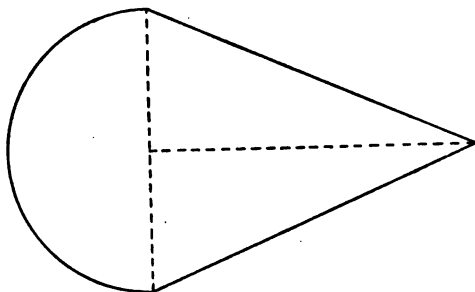


FIG. 4.

Find a formula for the area of a kite of the same shape whose breadth is a feet; and state what the area would be of a kite of the same shape and of breadth $3a$ feet.

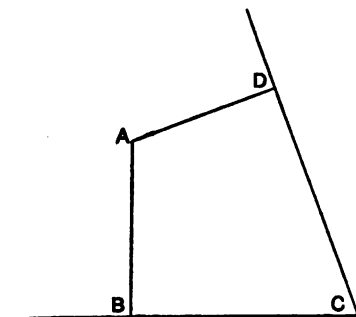


FIG. 5.

16. $ABCD$ (Fig. 5) is a plan (on a scale of 1 inch to 2 chains) of a corner plot of building land selling at £2200 an acre. Find the price of the plot. A chain is 66 feet.

17. The sides of a rectangle are measured as a inches and b inches. Write down (1) the area of the rectangle, (2) the distance

round the rectangle, (3) the difference between the lengths of the sides.

Give numerical values in each case taking a as 43 and b as 25.

If there is a possible error of x inches in each measurement, so that one side of the rectangle may have any value between $(a+x)$ and $(a-x)$ inches, and the other side any value between $(b+x)$ and $(b-x)$ inches, what is the greatest possible error in the area? Give the numerical value when $a = 43$, $b = 25$, $x = 0.5$.

18. Measure and weigh a round ruler or other cylinder, and find, and show in a table, its length, diameter, and volume, and the weight per cubic centimetre of the wood it is made of.

19. An isosceles triangle has the two sides each equal to 5 cm., while the vertical angle increases from 0° to 90° . The table below gives the length of the base for certain positions. Give values to fill up the two gaps in the table, and show the changes in the length of the base in a graph, taking 1 inch horizontally to represent 10° , and 1 inch vertically to represent 1 cm. From the graph read off to the nearest degree the size of the vertical angle when the base is 3.6 cm., and to the nearest millimetre the length of the base when the vertical angle is 76° .

Vertical angle	10°	20°	30°	40°	50°	60°	70°	80°	90°
Base, in cm.	0.87	1.74	2.59	3.42	4.23		5.74	6.43	

20. At 12 noon a boy begins to walk along a road at 4 miles an hour, and at 2 p.m. a cyclist rides after him at 10 miles an hour. Show that at the end of t hours after noon the boy has gone $4t$ miles and the cyclist $10t - 20$ miles. At what time will the cyclist overtake the boy?

Show in a graph the distances travelled in any time by the boy and the cyclist, and use the graph to find when the cyclist overtakes the boy.

CHAPTER VII

ON CRANES: PROBLEMS IN THREE DIMENSIONS

1. This chapter will treat solid geometry from the experimental and intuitional standpoint in order to bring out and add to the subconscious experience on which geometrical knowledge rests. Then we shall return to two-dimensional questions and so give the three-dimensional ideas time to arrange themselves before the logical discussion of them is undertaken.

Special care with regard to the rate of advance is necessary here. For the best result with regard both to knowledge and to training the method should be mainly that of discovery by the pupils, and this is possible only with a slow advance. If the advance is quicker the teaching can only be dogmatic. Still greater speed will make the pupil's mind merely a confused blank, and may even be a danger to health.

Wall crane.

2. A simple form of crane is the wall crane (Fig. 1). BA and CA are two rods fixed rigidly to a wall at B and C and to one another at A . The load L is raised by means of a rope passing over a pulley at A . Make a model crane of bamboo rods and find experimentally how the load moves as the rope AL is lengthened or shortened.—It moves in a straight line.

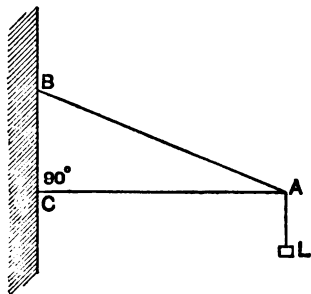


FIG. 1.

A wall crane is sometimes hinged at B and C so that it can turn round till it lies flat against the wall. If the

length of rope AL is kept constant while the crane turns, how does the load move? To find out arrange the length AL so that the load nearly touches the floor, and mark its various positions on the floor. And how does the point A move?—By experiment both paths appear to be circles.

3. How does the load move if the crane turns and the rope is varied in length at the same time? How does the point A move if the rod AB is removed? How does A move if AB remains while AC is removed? In each of these cases the locus traced out is called a surface. Give a few phrases from everyday life in which 'surface' occurs.—The surface of a pond, the surface of the earth, the surface of the body, surface-deep.

The meaning of surface is not identical throughout these phrases, and in some is not very definite. A thing that is surface-deep affects to some slight depth the body to which the surface belongs; the surface of the earth may mean the top layer of the earth or it may mean where the earth and air meet; the surface of a pond may mean a top layer or skin of water, or it may mean where water and air meet. In mathematical usage the surface of the earth or of a pond means where the earth or the water meets the air.

In the case of the crane the locus of the load is hardly a surface in the mathematical meaning because the load has a certain size. When the load is very small the locus is more nearly a surface in the mathematical meaning. If we could have a load that did not measure anything in any direction, its locus as the crane turns and the rope varies in length would be a mathematical surface.

Is the wall of the room a surface? Is the paper on the wall a surface? What is the surface of the wall?

4. What path does the point A trace out as the crane turns? Lower the model crane till C and A are close to the floor (keeping BC upright) and see how A moves.— A moves in a circle.

What property of a circle has the path of A ?— A is always at the same distance from C . Is this enough to show that the path of A is a circle? Remove the rod BA ; then as CA turns A need not stay close to the floor. Does it still describe a circle? How does it differ from the circles we had in chapter I?—In chapter I a

circle was the path of a point that was always at the same distance from a tree and was always on the ground. In the present case A will describe a circle if it stays on the floor.

Is the ground a surface? Is the floor a surface?—No, but when a point moves on the ground it is moving in a surface, namely, in the surface where ground and air meet, it moves (in language that is at once general and mathematical) 'on the surface of the ground'. The floor is not a surface, it is about an inch thick; but 'the surface of the floor', on which the point A moves, is a surface.

5. Try to fit your straight edge against the floor; also against a ball; and against the side of a standard gallon or litre measure.—It fits against the floor in any position, it will not fit against a ball at all, in some positions it fits against a gallon measure and in others it will not. The surface of the floor is called a plane because the ruler fits in any position; the other surfaces are not planes.

Let two pupils stretch a string and try to fit it against the ground. On some ground it will fit in any position, and that ground (or rather the surface of it) is called plane. Other ground may be lumpy so that the string will not fit; such ground is not plane.

How is the word plain used in geography? (It is a pity such slight differences of meaning of the same word are marked by different spellings, plane and plain.)

6. When the bar BA of the crane is removed, and A moved to all possible positions, what locus does it trace out?—A surface, all points of which are at the same distance from C . Such a surface is called a sphere and C is called its centre. Can you name any objects of the same shape?—Cricket ball, association football, the earth.

When the bar BA is replaced and CA removed, how does A move? And what positions of A lie on this sphere and at the same time on the above sphere with centre C ?—All points of the circle on which A moves when the crane is complete lie on both these spheres.

Shear-legs.

7. The wall-crane is of use to raise bodies from the ground to upper stories. The shear-legs (Fig. 2) is used

for moving great weights horizontally as well as raising or lowering them. It consists of three beams DE , DF , DG

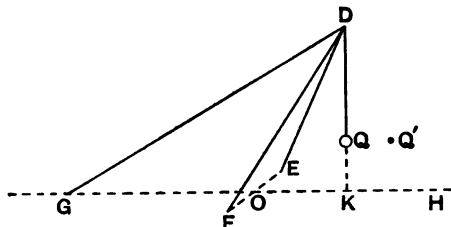


FIG. 2.

hinged together at D . DE and DF are equal in length and are hinged to fixtures on the ground at E and F . The end G of the piece DG is moved to and fro along a line GH that bisects EF at right angles. The load Q hangs by a chain passing over a pulley at D , so that the length DQ can be varied.

Make a model with bamboo rods and find by experiment how Q moves (1) when the length DQ changes, (2) when the length DQ does not change but G slides to and fro, (3) how D moves as G slides to and fro, (4) what motions should be given to the system to move the load from Q to another position Q' at the same height above the ground, (5) what is the locus of all points to which the load can be moved by combining the two motions. The discussion is helped by setting a piece of cardboard on its edge, the edge coinciding with the line GH . (1) is of course a straight line; (2) and (3) are circles; (4) advancing G will advance the load the necessary distance and lower it at the same time, so that the chain must be shortened to raise the load to the old level; (5) is a plane surface. Whether these experimental results are mathematically accurate will be discussed later.

Keeping the length DQ constant slide G to and fro, and in various positions measure the height QK of the load above the ground and the distance OK by which Q is in front of O . Plot these on a sheet of paper to show the motion of the load.

Find the motion of D in the same way.

The load on the shear-legs has two motions, one by sliding

G to and fro, the other by lengthening or shortening the chain; we say that the load has two degrees of freedom. In the case of the wall-crane, when the crane is fixed and will not turn flat against the wall, the load has one motion only, it moves only up and down. In this case we say the load has one degree of freedom.

8. Measure the distance DO in several positions. What do you find? Can you prove that the distance DO is always the same? Measure in several positions the angle that DO makes with EF . Can you prove this angle constant?— E and F are fixed points so that the distance EF does not change, and the lengths DE and DF (the lengths of the beams) of course do not change; so that the triangle DEF is fixed in form and only changes in position; and DO , the line joining a vertex to the middle point of the opposite side is fixed too. So, in our model, we could replace the rods DE and DF by a pasteboard triangle DEF . And O being midway between E and F , OE is equal to OF , while the lengths of the beams DE and DF are the same; so that the triangles DEO and DFO are congruent, that is, they would fit if folded together; so that $\angle DOE$ and $\angle DOF$ are right, and DO is perpendicular to EF .

Do you know enough to prove that D moves in a circle?— D is always at the same distance from O ; but this is not enough, the locus of all points at the same distance from O is a surface that is called a sphere. We know further that the constant length OD stands out always at right angles to EF . It may be that a line of constant length turning about another and being always perpendicular to it traces out a circle with its extremity; but it is not yet proved. Notice that in the wall-crane the locus of A (Fig. 1) is that of the end of a rod CA that turns about BC and is always perpendicular to it.

9. If the bar GD is removed, how does D move?—As before. If the bar DE also is removed?— D moves on a sphere. If the bars DG and DF were removed, how would D move?

10. Suppose a shear-legs has GD 8.6 metres long, DE and DF 5.8 metres long, and the feet E and F set 8.6 metres

apart. Find the distance DO by drawing the triangle DEF (Fig. 3, which is only a sketch) on a scale of $\frac{1}{100}$. Now (Fig. 4, sketch) use this distance and the known length GD to find where G must be when the shear-legs is to pick up a load 2 metres in front of O , and what length of chain is wanted to reach down to this load.

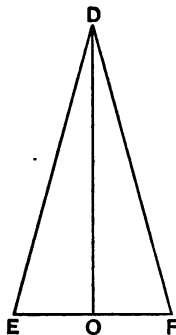


FIG. 3.

If the load is to be set down again 3 metres in front of O , how far forward must G move? And if it is to be set down on the deck of a ship 3 metres in front of O and 1 metre below the ground-level, how long must the chain then be?

Give measurements of the position of the load that you could take so as afterwards to replace the load in its position.

—We could give the distance in front of O and the distance above or below ground-level; or the angle that OD makes with

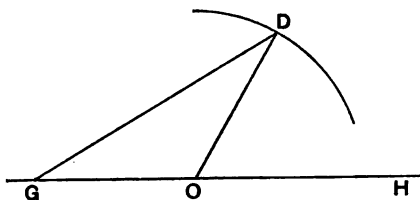


FIG. 4.

OH and the length of the chain; or the distance of G behind O and the length of the chain; or other sets of measurements.

Traveller crane.

11. Another form of crane is the traveller crane sometimes used in an engineer's yard. Figs. 5 and 6 show two sketches of this crane, Fig. 5 after the manner of a picture; Fig. 6 shows it as seen from a greater distance, and is technically termed a projection. $ABCD$ is a rectangular frame supported on 4 pillars. A beam EF rests on 2 sides of the rectangle and travels along them, remaining

always parallel to the other 2 sides. The crane G travels to and fro along EF and raises or lowers the load H . A bar across a box without lid, and carrying the load by a string, will serve as model. Or two desks may be placed so that

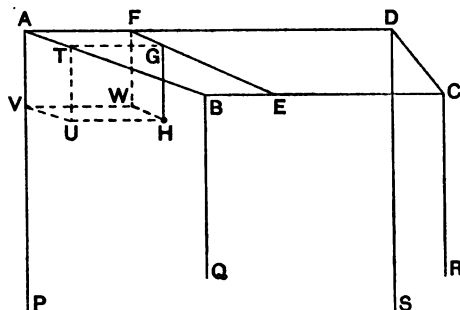


FIG. 5.

their edges will serve as the sides BC and AD of the frame, to carry the bar EF .

How does the crane move when the beam travels? How does the load move when the chain is lengthened, when the

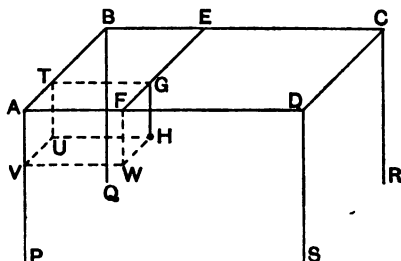


FIG. 6.

crane travels, and when the beam travels? How may the load move when any two of these motions take place at the same time? How when all three motions take place together?—The locus for any single motion is (experimentally) a straight line, for any two motions a plane, and for all three motions at the same time a block of space. When

the three motions take place at once the load may reach any point of space under the rectangular frame.

12. How would you move the load to a given spot?—Get the beam over the spot, then move the crane along the beam till it is over the spot, then raise or lower the load to the given spot. Or in any other order, for instance, bring the load to the right level, then the crane to the right distance from an end of the beam, and finally the beam over the spot.

What measurements would you take of the position of the load so as to be able to replace it?—Take the distance of the beam from the end AB of the rectangle, the distance of the crane from E , and the length GH of the chain. How many degrees of freedom has the load?—Three, which (supposing AB to point northwards) we may describe as (1) north-and-south motion, (2) east-and-west motion, (3) up-and-down motion.

If we take away one degree of freedom (or, as it may be expressed, if we impose one degree of constraint) by preventing the crane from travelling along the beam, what freedom is left?—Two degrees of freedom are left, the east-and-west motion and the up-and-down motion.

If we impose a second degree of constraint by keeping the length of the chain fixed, what freedom is left?—One degree only, the east-and-west motion.

And if we impose a third degree of constraint by forbidding the beam to travel?—No freedom is left, the load cannot move at all.

13. How does the crane G move when the beam EF travels? Refer to the explanation of parallel lines in chapter II, article 2.— G moves in a straight line parallel to BC and at a distance EG from BC .

How do you know?—Because EG has always the same length and is always perpendicular to BC .

Is this enough? Would the path of G still be a straight line parallel to BC if the end F of the beam did not rest on AD , but EF turned about BC (while E travels along BC), so that EF sometimes pointed up, sometimes down, and sometimes in other directions? Find by experiment.—It is not enough. The path of G is not then straight.

What further condition is necessary, and what further condition was implied in chapter II?— G must remain in the plane $ABCD$. In chapter II all the

points and lines were assumed to lie on the sheet of paper, that is, in the same plane.

You have seen experimentally that when the crane G travels the load H moves in a straight line. Can you describe the position of this line?—It is parallel to EF and at the distance GH from it. For the length of HG is the same all the time, and (experimentally) HG is always $\perp$ to EF , and lies all the time in a certain plane.

14. When the beam and the crane travel at the same time how may the crane move?—It may move to any point in the plane $ABCD$.

And how may the load move, the length GH remaining constant?—It also moves in a plane, and the distance GH between the plane in which H moves and the plane in which G moves is the same everywhere.

What angles does GH make with lines in these two planes drawn from G and from H ? To measure the angles made with lines in the plane $ABCD$ lay a board over the model to cover $ABEF$ or $EFDC$; to measure the angles made with the plane H moves in, take the string GH of such a length that H almost touches the bottom of the box.—The angles are all right angles.

When a number of lines are drawn in a plane through a point G (Fig. 7) and another line GH stands so as to be perpendicular

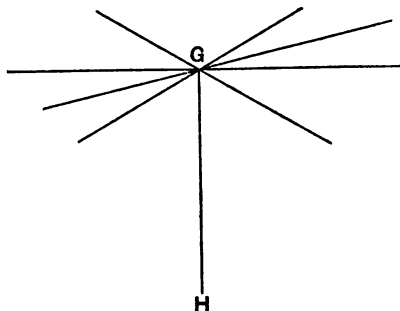


FIG. 7.

to all these lines, the line GH is said to be perpendicular to the plane. And when a number of lines (such as GH) are perpendicular to a plane at different points, and all have

the same length, the plane in which the ends of these lines lie is said to be parallel to the first plane. So that, in the present case, the locus of the load H (when the beam and the crane travel) is a plane parallel to the plane of the frame $ABCD$.

15. Give examples of parallel planes from the school or a house.—The floor and the ceiling are parallel planes, the north and south walls are parallel planes, so are the east and west walls. All the floors in the school or house are planes parallel to one another. Of course 'floor' means here not the boards which may be 2 or 3 cm. thick, but their upper surface on which we walk; 'wall' does not mean the stone or brick mass 30 or 40 cm. thick, but the surface of it that we see; and so with 'ceiling'.

The line in which the rope or chain of a crane hangs is called a plumb line or vertical line. Any rope or chain hung by one end hangs plumb or vertical. A plane perpendicular to a plumb or vertical line is called a horizontal plane; thus the floor, the ceiling, and the rectangle $ABCD$ of the traveller crane are horizontal. A plane against which a plumb line will hang (without pressing against it) is called a vertical plane; the walls of a room are vertical planes.

16. We saw in Article 12 that we can fix the position of the load H (Figs. 5, 6) by stating how far the beam FE must move from the position AB , how far the crane must move from F , and what length of chain must be paid out, to bring the load to the position in question. And the order of these motions is indifferent.

In more mathematical language we may say that the position of the load is fixed by its distances from three planes, namely, the distance AF or BE from the plane $ABQP$, the distance FG from the plane $ADSP$, and the distance GH from the plane $ABCD$. Or, again, we may say that a point can travel from A to H by going suitable distances in the direction of (that is, along or parallel to) the three lines, AB , AD , and AP , namely, a distance AF in the direction of AD , FG in the direction of AB , and GH in the direction of AP . And the order of these movements is indifferent; thus the route may be along $AFGH$ or $AFWH$ or $ATGH$ or $ATUH$ or $AVUH$ or $AVWH$.

17. Arrange your model so that the distances travelled from A by the load are 80 cm. in the direction of AD , 50 cm. in the direction AB , and 40 cm. in the direction AP . Measure how far the crane would have had to travel in a straight line to get from A to its present position, and how far the load would have had to travel in a straight line to get from A to its position. Can you test these distances by drawing? —We lay down (Fig. 8, sketch) the length AF , let us say on a scale of $\frac{1}{2}$, so that we make AF 16 cm. long; and draw FG

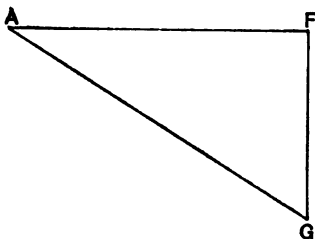


FIG. 8.

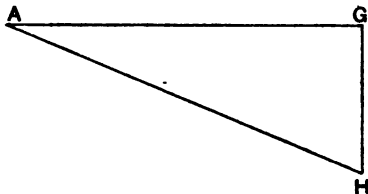


FIG. 9.

at right angles to it and 10 cm. long, to represent FG on the same scale. We then measure AG ; it is 18.9 cm. long, representing a distance on the model of 94 cm. We now lay down $AG = 18.9$ cm. (Fig. 9, also a sketch) and draw GH at right angles 8 cm. long to represent 40 cm. We then measure AH and find it 20.5 cm., so that the distance of the load from A in the model is 102 cm.

Place the beam, crane, and load at random. Specify the position of the load by the distances AF , AT , AV (Figs. 5 and 6). Measure the distances of the crane and the load from A , and check by drawing as before.

Scotch crane.

18. The Scotch crane (Fig. 10) has a movable part that consists of the vertical post AB , the jib AC , and the tie-rod BC . The post AB is hollow and a pillar or pivot fixed to the ground fits into the hollow, so that the crane may revolve about this pivot. The load hangs from C by a chain that may be varied in length. AD is a platform that carries the steam engine and tackle for working the crane.

What path is described by the load as the chain shortens or lengthens? What path by the load and by the point C when the crane revolves? Discover by experiment with

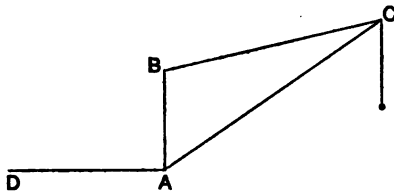


FIG. 10.

a model in which the post is 40 cm. long, the jib 100 cm., and the tie-rod 80 cm..

From what points on the ground can this crane pick up a load? What is the radius of the circle on which these points all lie? Find from measurement of the model, and also by drawing; use for the drawing $\frac{1}{10}$ of the scale of the model.—We draw the triangle made up of post, jib, and tie-rod, making the post BA perpendicular to the ground DA . The chain will hang vertically, that is, parallel to the post;

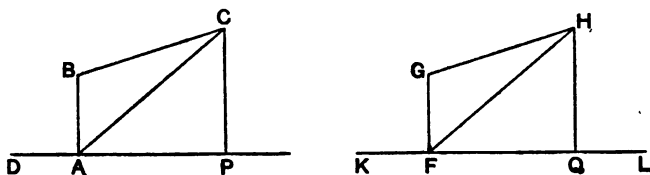


FIG. 11.

so by drawing $CP \parallel$ to BA we find the position P (Fig. 11) from which the crane will pick up a load.

19. Is the distance of P from A the same for all positions of the crane?—By experiment with the model it is. By drawing the crane for other positions we find always the same length AP , namely, 7.6 cm. General reasoning gives the same result. For suppose FGH to be another drawing of the crane, Q the point where the chain meets the ground KFL . The two triangles ABC and FGH have their sides equal in pairs, and so will coincide if fitted together. The

angles GFL and BAP are both right angles, and so will coincide. We thus have FH coinciding with AC and FL lying along AP . Further, we know (chap. V, art. 18) that two triangles ACP and FHQ having $AQ = FH$, $\angle CAP = \angle HFQ$, and the angles at P and Q , right angles, will coincide. So that $FQ = AP$; or the point where the chain meets the ground is always at the same distance from the pivot. Repeat the reasoning by which you know the triangles FHQ and ACP to be congruent.

Ex. 1. For a crane that has a jib 10 metres long and inclined at 40° to the post, find, by drawing, the radius of the circle from which loads can be picked up.

How many degrees of freedom has the load? How many has the jib-head C ? The surface on which the load may move is called a cylinder.

20. The Scotch crane is sometimes mounted on a truck, which runs along rails, and gives a greater choice of positions from which loads can be picked up. Suppose a crane, whose jib is 9 metres long, and makes an angle of 43° with the post, to run on a straight railway track 80 metres long. Find, by drawing, the locus of possible positions on the ground for the load when the truck is still.—It is a circle of radius 6.1 metres. By considering this locus for all possible positions of the truck, find and draw the locus of all possible positions of the load on the ground,

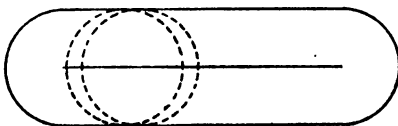


FIG. 12.

scale $\frac{1}{200}$.—It is a straight piece of ground 12.2 metres wide ending in two semicircles (Fig. 12).

21. When the truck is at rest and the jib swings round, what is the locus of the jib-head?—It is a circle of radius 6.1 metres, at a height of 6.6 metres above the ground.

If at the same time the length of chain by which the load hangs varies, to what positions can the load

be brought?—It can be brought to any point vertically below a point on this circle, that is, to any point on the part of a vertical cylinder between this circle and an equal circle on the ground.

If at the same time the truck travels on the rails, to what positions can the load be brought?—It can be brought to any point above the area shown in Fig. 12 not more than 6·6 metres above the ground. Its locus is a block of space roughly of the shape of a date-box, bounded by the area shown in Fig. 12, by an equal area 6·6 metres above it, and by the vertical lines joining these two areas.

Make a model in clay or plasticine of this block of space.

Derrick crane.

22. The derrick crane has the post AB (Fig. 13) held in position by two other beams or stays AE and AD fixed to the ground at E and D . Further, the jib BC is supported by a chain AC which can be altered in length and so change the angle of inclination of the jib to the vertical. Suppose the crane-post AB to be 10 metres long and the jib 18 metres. E and D are 10 metres from B and the angle EBD is a right angle.

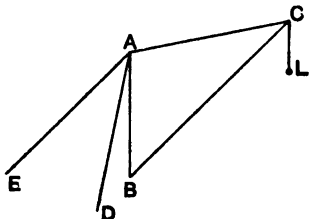


FIG. 13.

By drawing the triangle ABD to a suitable scale, say having $AB = 10$ cm., $BD = 10$ cm., $\angle ABD = 90^\circ$, find how long AE and AD must be made. Use this result to make a bamboo model.

Use the model to find the locus on the ground of possible positions of the load when the chain AC is 18 metres long.—It is part of a circle of radius 12·7 metres, $\frac{3}{4}$ of the circle, because the stays AE and AD keep the jib from swinging right round. Check the radius of this circle by drawing.

What is the inclination to the vertical of the jib when the chain AC is 18 metres long? Find from the model and by drawing.

23. Find, from the model and by drawing, the locus of possible positions of the load on the ground when the jib makes an angle of 28° with the post, and when it makes an angle of 80° . What is the length of the chain AC in each case? If the angle between the jib and the post may be anything between 28° and 80° what positions on the ground are possible for a load? Show the positions on a drawing. —The positions all lie between two circles with centre B

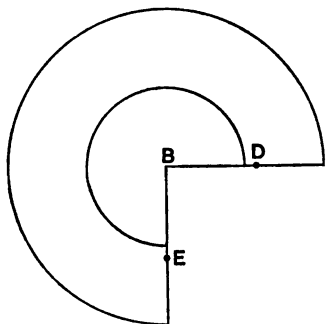


FIG. 14.

and radii 17.7 and 8.5 metres, excluding the part bounded by the lines BE and BD (Fig. 14).

When the length of AC is 13 metres, what positions may the load have for different lengths of the chain CL ? and for different positions as the jib swings round, the length CL being unaltered?

When the length CL remains unaltered, and the jib does not swing, but the length AC is altered, how

does the load move? and how does the jib-head C move?

How many motions (or degrees of freedom) has the jib-head C ? When these 2 motions take place what is the locus of the jib-head?—A sphere of radius 18 metres and centre B , or rather the half of it above ground, excluding the part cut off by the stays AD and AE .

And if the angle between the jib and the post must lie between 28° and 80° , what is the locus?—Part of the half-sphere at the top and part at the bottom are then impossible. A certain Japanese lamp-shade gives the form if we cut away the quarter made inaccessible by the stays AD and AE .

24. How many degrees of freedom has the load? What is the locus of the load for each motion?—When the length CL changes the locus is the part of a vertical straight line between the jib-head C and the ground. When the jib swings round, the locus is $\frac{3}{4}$ of a circle, the remaining quarter being cut off by the stays. When the jib is raised

or lowered, the locus is the part of a certain circle lying above the ground and on one side of the crane-post.

What are the possible positions for the load when two of these motions are allowed at the same time?—When the jib is raised or lowered and the length of chain CL altered, the jib-head describes a quarter of a circle, and the load may be anywhere vertically below a point on this quarter-circle. Its locus is the part of a vertical plane bounded by

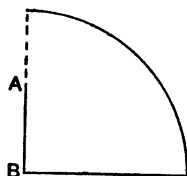


FIG. 15.

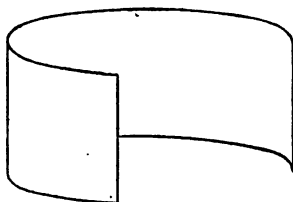


FIG. 16.

this circle, by the crane-post AB and a prolongation of it, and by the ground (Fig. 15). When the jib swings and the length CL changes, the jib-head describes $\frac{3}{4}$ of a horizontal circle, and the load may be anywhere below a point on this $\frac{3}{4}$ -circle. Its locus is the part of a cylindrical surface sketched in Fig. 16. When the jib swings and is also raised or lowered, the jib-head describes part of a sphere, namely the



FIG. 17.

half above ground excluding the part cut off by the stays. The load copies this surface, at a distance below equal to the length CL , so far as the copy is above ground (Fig. 17).

What are the possible positions of the load when all three motions are allowed?—The jib-head may be anywhere on a certain half sphere except the part cut off by the stays, and the load anywhere vertically below the jib-head. So that its locus is the block of space shown in

Fig. 18. Model this block of space in clay or plasticine.

And if the angle between the jib and the crane-post must lie between 28° and 80° , what is the locus? For the locus we must cut away from Fig. 18 all that lies within the

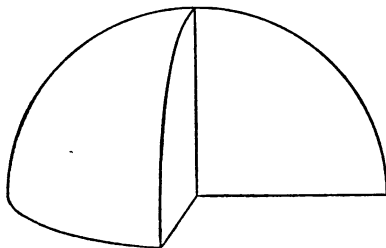


FIG. 18.

vertical cylinder standing on the circle of radius 8.5 metres (on our reduced scale), and all that lies outside the cylinder that stands on the circle of radius 17.7 metres (Fig. 19).

Model this block of space.

25. Suppose the jib to make an angle of 40° with the crane-post, and to lie at first against the stay AD and then

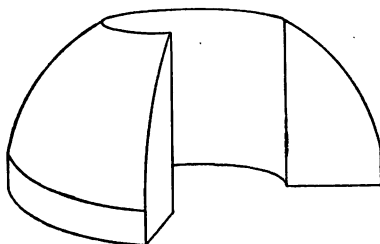


FIG. 19.

to swing round till it is in the same plane as the stay AE . Suppose the load L to hang just clear of the ground all the time. Draw a figure of the ground showing the feet $B D E$ of the crane-post and the stays, and showing the position of the load when $\angle DBL$ is equal to 30° (Fig. 20).

A sort of picture of the crane may be made by setting

up a sheet of paper against AB and AE , and drawing from each point of the crane a perpendicular to this plane. The

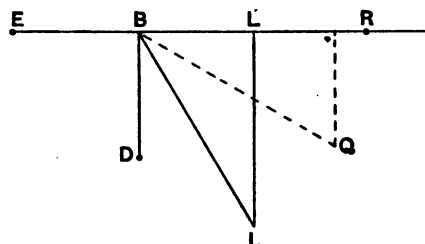


FIG. 20.

foot of the perpendicular represents the point from which it is drawn. Draw this picture.—The perpendiculars from points on the stay AD fall on the crane-post so that one line AB (Fig. 21) serves for this stay and the post. L being

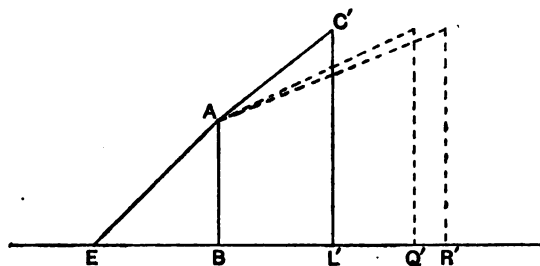


FIG. 21.

on the ground we can draw the perpendicular LL' in Fig. 20, and so find the point L' that represents the load in our picture. The line CL will have the same length in the picture as in its actual position. By drawing (Fig. 22) the jib at 40° with the crane-post we find the length of CL to be 13.8 metres; so we draw (Fig. 21) at L' a line $L'C' \perp$ to EBL' and showing a length of 13.8 metres on the scale we are using. Now by joining A to C' we finish our 'picture'. Such a picture is called a projection.

Call Q the position of the load when it has swung round through 60° from BD , R the position when it has swung

through 90° , and make pictures of these positions. These pictures are shown in Fig. 21 by dotted lines.

Co-ordinates.

26. It was seen (Arts. 12 and 16) that the position of the load on a traveller crane can be specified by giving its distances north, east, and downwards from the corner *A* (Figs. 5, 6). The load can be moved only to points in the block of space enclosed by the frame that supports the crane; but if we wish merely to specify a particular position and not to move the load to that point, we need not restrict ourselves to that block of space. We may state the position of any point by giving its distance north or

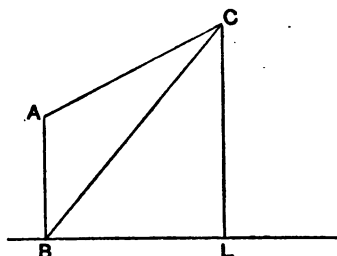


FIG. 22.

south of *A*, its distance east or west of *A*, and its distance above or below *A*. When the position of a point is given in this way these distances are called co-ordinates, the lines *AB*, *AD*, *AP* are called co-ordinate axes, and the point *A* is called the origin of co-ordinates. Of two opposite directions, e.g. north and south, it is usual to choose one, say north, and to agree that a distance stated simply, without mention of north or south, is to be measured north; while a distance to be measured to the south has a minus sign attached. Thus if a point has a co-ordinate of 3 metres for the axis *AB*, the point is 3 metres north of *A*; another point with a co-ordinate -3 metres is 3 metres south of *A*.

27. Again, suppose that a derrick crane has a jib (*BC*, Fig. 13) that can be shortened or lengthened as much as

we like, and suppose the jib may swing right round past the stays AD and AE , and that the jib can be raised till it points vertically upwards, and lowered till it points vertically downwards. This arrangement would enable the jib-head C to be brought to any position whatever. It would not make a very serviceable crane, but it serves very well for stating the position of a point. Suppose the jib-head to be at the point to be specified. Then if we measure the length of the jib, the angle the jib makes with the crane-post BA , and the angle through which the crane must swing to come against the stay AD , we have measurements enough to enable us to put the crane back in this position whenever we like; that is, we have measurements enough to fix the position of the point. These three measurements (a length and two angles) are another system of co-ordinates.

The cranes we have discussed suggest various other ways of fixing a point, and any system of measurements will do for a system of co-ordinates. But the two just mentioned are more used than any others.

28. When we are concerned only with points in one plane simpler systems of co-ordinates are possible. Thus the position of the traveller crane may be stated as so far north of A and so far east of A (Figs. 5, 6). And any point in the plane of the frame $ABCD$, whether accessible to the crane or not, may be specified by its distances north or south and east or west of A .

Again, if our imaginary derrick crane is not allowed to swing round, the jib-head may still be brought to any position in a certain vertical plane, and we may specify this position by the length of the jib and the angle it makes with the crane-post.

To make a plane surface.

29. Your desk has a plane surface, and so has a book. Place the book on the desk. Do the two plane surfaces touch all over? Slide and turn the book about. Does it fit against the desk in all positions?

If you take two rough pieces of sandstone and place them together, do they fit? If you rub them together till the irregularities are worn away and they fit, what sort of sur-

faces have they?—If one is rubbed to and fro on the other they may come to fit as the axle of a wheel fits into the wheel. But we can prevent that by turning one of them about, and rubbing till they fit in all positions.

Do you know any other surfaces than planes that fit in all positions? Do an egg and egg-cup fit in all positions?—They fit if we only rotate the egg without trying to turn it upside down. We can turn it upside down only by taking it out of the cup. So that they do not fit in all positions.

Do the cup and ball used in the game called 'Cup and Ball' fit in all positions? If a ball is pressed into a heap of sand so as to make a hole that it fits, will it fit in all positions?—Yes, in both cases.

So that if two pieces of sandstone are rubbed together till they fit in all positions it might happen that one had a surface like a ball, a spherical surface, while the other had a hollow surface to fit it, also a spherical surface. If we took three pieces of sandstone and rubbed them till they fitted two and two in every position, could their surfaces be spherical?—No, for when two spherical surfaces fit, one is round like a ball and the other hollow, so that these cannot both fit the third one.

What kind of surfaces will these three pieces of sandstone have?—They *may* have plane surfaces.

As a matter of fact, when a true plane surface is wanted, three of them are made at once in this way. Three plates of metal are taken and ground away till their faces fit, two and two, in all positions.

Having three plane surfaces made in this way, how could you make a straight line?—If one of the plates is cut across, and then the side ground till a second plate fits on the side just as it does on the face, the edge where face and side meet is a straight line.

EXERCISES.

2. In a room whose walls run north and south, and east and west, an electric glow lamp is to be placed at a certain point. The wire enters the room at the north-east corner of the ceiling, and is to run everywhere parallel to an edge of the room. (The line where two walls, or a wall and ceiling, or a wall and floor meet is an edge.) Show how the wire might be run actually, and what other ways are possible if you need not trouble about supporting

the wire or about the convenience of people walking about the room.

What co-ordinates would you use to specify the position of the lamp?

3. How many degrees of freedom has a railway train? How many has a steamship? And an airship?

4. A derrick crane has a jib 20 metres long and post 9 metres long. When the chain AC (Fig. 13) is 17 metres long it picks up a load, and then without change of the length CL the chain AC is shortened to 15 metres. Find by drawing through what height the load is raised.

The crane picks up a load from the ground when the jib makes an angle of 50° with the vertical. The jib is then raised till it makes 35° with the vertical, then the crane swings round through 20° , and deposits the load by lengthening the chain CL by 2.6 metres. What is the difference in height between the first and last positions of the load?

5. In building a bridge materials are sometimes brought into position by running them along a cable whose two ends are made fast to posts on the two banks. Suppose the tops A and B of the posts 30 metres apart and the cable 34 metres long, and suppose each part of the cable from the load W to a post to be straight. Find by experiment with a model and by drawing, the path the load describes as it travels along the cable.

The load will of course always be between and below the points A and B , and in the vertical plane through A and B . But imagine it to be able to take all positions in the vertical plane the sum of whose distances from A and B is 34 metres. Draw the locus of all points it could then reach. This locus is called an ellipse.

Again, suppose the load not confined to this vertical plane, but able to reach any position in or out of the plane the sum of whose distances from A and B is 34 metres. Investigate the form of the surface it could describe, and make a model of the surface. This surface is called an ellipsoid. If the load is made fast to a point of the cable, so that its distances from A and B are always the same, what kind of path can it describe?

6. A board has cut in it two grooves crossing at right angles. A rod AB has on it two knobs at A and B which slide along the grooves. Investigate the kind of path any point P of the rod describes. For this purpose draw two straight lines at right angles to represent the grooves, and draw on tracing-paper a line with the knobs A and B marked to represent the rod.

Draw the path described by P for a number of different positions of P on the rod, some between A and B and some outside. For

what positions of P on the rod are the ovals flatter and for what positions are they rounder? All these ovals are called ellipses.

7. A shear-legs (Fig. 2) has a backstay GD 9 metres long, the head D moves in a circle of radius 7 metres, and the foot of the backstay may not come within 3 metres of O (the centre about which D turns), nor may it go further from it than 7 metres. Find how much of its circle D may describe, and where a load must lie for the shear-legs to be able to pick it up.

8. A plumb-line hangs from a point A on the ceiling of a room ;

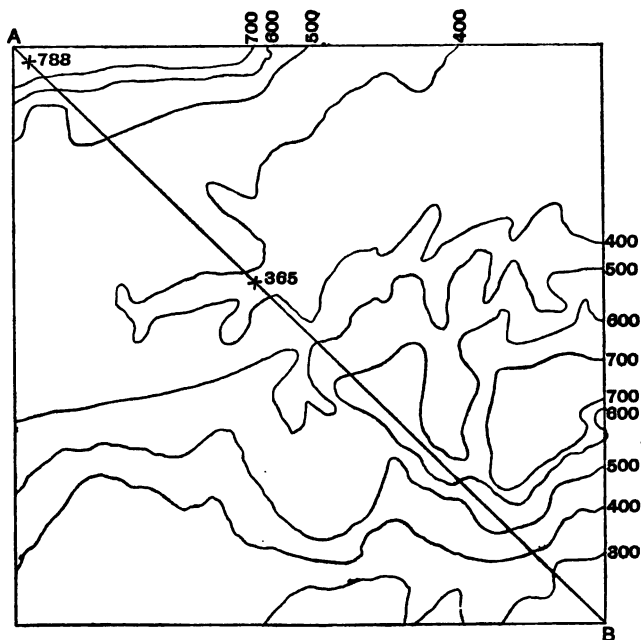


FIG. 23.

two knots on the line, respectively 7.6 feet and 3.2 feet from the floor BC , have their shadows thrown by an arc lamp at distances of 12.35 feet and 1.9 feet from the point where the line meets the floor. How high is the lamp above the floor, and how far is it from the plumb-line? Find out by drawing on a scale of 1 cm. to 1 foot.

9. Suppose you have a triangle cut out of paper. Suppose it bent. Is it still a triangle? Why do you say so?

Suppose the triangle placed on your desk, and the upper side marked with a cross. By what measurements could you determine its position so that if the triangle is removed you could replace it exactly in position? Do not give more measurements than you think are necessary, and explain why you consider that those you give are sufficient.

10. The map with contour lines (Fig. 23) is drawn on a scale of an inch to a mile, each contour line showing the height above sea level of the land through which the line passes. The numbers placed at the ends of the lines or at special points denote the height in feet. You are to draw a curve showing the rise and fall of the land along the line AB . Work on squared paper representing the height on a scale of 1 inch to 200 feet, and the distance along AB on a scale of 1 inch to a mile.

11. To find the height of a hill-top O above a plain two points A and B 200 metres apart are taken on the plain and in the same vertical plane with C . The angles of elevation of C at A and B (that is, the angles CAK and CBK , where K is a point on AB produced) are measured and found to be 23° and 36° . Find the height of the hill-top by drawing to a suitable scale.

12. A pyramid $OABCD$ stands on a horizontal square base $ABCD$ 7 cm. in the side. The vertex O is 5 cm. vertically above the point P , where the diagonals AC and BD of the base meet. Find the length AP by drawing the base, then find the length OA by drawing the triangle OAP , and so draw one of the triangular faces of the pyramid.

Measure the angle between AO and AP , and the angle between AO and AB .

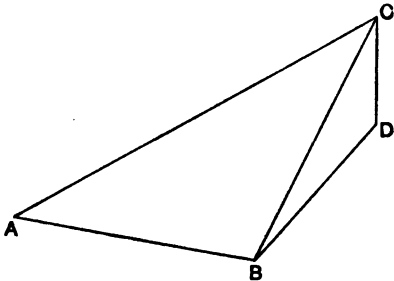


FIG. 24.

13. To find the height of a mountain C (Fig. 24) above a plain two points A and B are taken in the plain 1000 metres apart. The angles CAB and CBA are measured and found to be 62° and 84° . Also, CD being vertical and BD horizontal, the angle CBD is measured and found to be 12° . By drawing the triangle ABC to a suitable scale find the length of BC , and then by drawing the triangle BCD find the height CD of the mountain.

CHAPTER VIII

THE ANGLE-SUM PROPERTY OF A TRIANGLE. PARALLELS AND AREAS

1. In chapter V we saw that when two angles of a triangle had been made we had no choice about the size of the third angle, we saw that there was some relation among the three angles. We will now discuss the nature of this relation. A knowledge of this relation may help us to make a triangle from such data as $a = 6.9$ cm., $B = 80^\circ$, $A = 39^\circ$. Let the pupils use the methods of chapter V to make this triangle; they will notice that they are clumsier than if we knew a , B , C .

2. Make a right angle by folding a sheet of paper and then folding the crease on itself. Cut along the creases, and from one of the four pieces of paper cut a large right-angled triangle ABC (Fig. 1). Fold the corners A and C on to the corner B , B being the right angle. How do the angles A and C now lie? Does this experiment tell you anything about the size of the two angles together?

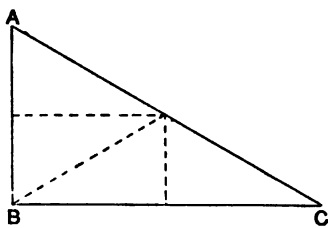


FIG. 1.

In this and the following experiments each pupil should make his own triangle. The triangles will thus be varied in shape, and the evidence of more value than if all used triangles of the same shape.

3. Now let each cut out a paper triangle of any shape. You know already that one side of a triangle is often chosen

out and called the base. When one side has been chosen for base, the angle opposite to it is called the vertex. Choose a base in each triangle, in such a way that the obtuse angle of triangles that happen to be obtuse-angled is the vertex, and call the vertex A . Fold the base BC on itself in such a way that the crease passes through A (OA in Fig. 2).

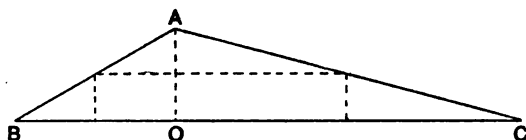


FIG. 2.

Now open out the triangle and fold all three corners on to O . What do you observe? Does the experiment tell you anything about the sum of the three angles?

Now cut along the crease OA , and what have you?—Two right-angled triangles, each giving the case of Art. 2 over again.

4. Let each cut out another triangle of any shape. Tear off the three corners, and place the angles side by side as in Fig. 3. What do you observe? What does it tell about the sum of the three angles?

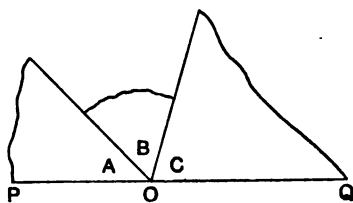


FIG. 3.

All these results are subject to the drawbacks inherent in experiment, one of which is that the result may be true for the particular triangle dealt with, but not for other triangles. This objection is to some extent met by the variety of triangles used, so that we know that the result is true for a considerable number of particular triangles at least. A more serious drawback is the unavoidable error in drawing, folding, measuring, and judging the fitting of the angles. Even if we could be sure there was no error of a tenth of a degree, we should still be in doubt whether there might

124 THE ANGLE-SUM PROPERTY OF A TRIANGLE

be errors of a hundredth of a degree. The following discussion is less subject to these drawbacks.

5. Let the whole class now draw triangles, each as he likes. Let each measure the three angles of his triangle, add them up, and report the result. These results will not differ much, except that one or two may differ from the rest by ten or twenty degrees or more, and are clearly due to blunders. Let these latter be discarded and the average of the remainder calculated.

This average is likely to be nearer the true value than any individual result. For some individual results will be too great and some too small, and one kind of error is as likely as the other, so that when we take the average we may expect the errors to cancel one another to some extent.

6. A pavement is sometimes made up of a number of triangular tiles, all of the same size and shape, that is, all congruent to one another. Look at the drawing of such a pavement in Fig. 4, in which the angles of one triangle

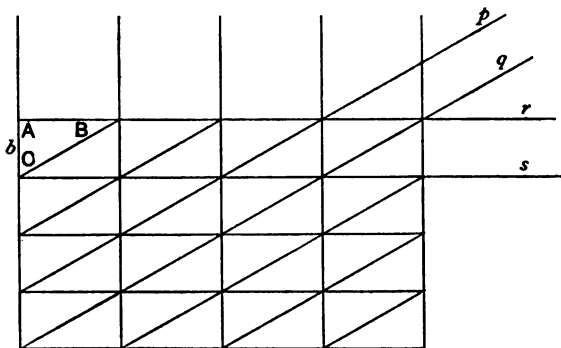


FIG. 4.

are marked *A B C*. Mark all the other angles of the pavement that are equal to *A*, all those equal to *B*, and all those equal to *C*. How many of each kind are there at a corner where the tiles join? And through what angle do you turn if you first face in any direction and then turn till you

again face in the same direction? What then is the sum of A , B , and C ?

If the pavement is indefinitely extended, will the distance between the lines r and s vary, or the distance between the lines p and q ?—They do not vary; the distance between r and s is the length of the side b of the triangle ABC ; the distance between p and q is the height of the triangle when a is considered the base. What are such lines called?

What difference would there be in the results of this article if the tiles were not right-angled?

7. Stand up facing north, then turn round till you again face north. Through how many degrees have you turned? Stand against the middle of the east wall of the room, facing north; then walk along the walls till you arrive again at your starting-point. Through what angle have you turned altogether, and through what angle did you turn at each corner of the room?

If you start from the point K of the triangle in Fig. 5,

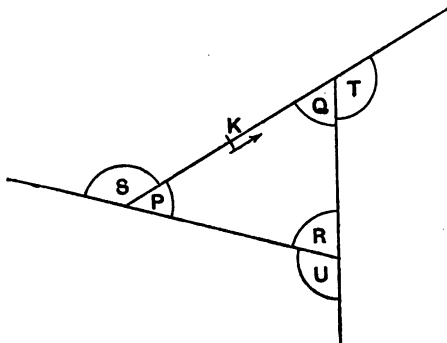


FIG. 5.

and walk in the direction of the arrow round the triangle and back to K , through what angle have you turned altogether, and through what angle did you turn at each corner? What then is the sum of the angles $S + T + U$? What is the sum $P + S$? $Q + T$? $R + U$? What is the sum $P + Q + R$?—Since $S + T + U = 360^\circ$ or four right angles, and $P + S + Q +$

126 THE ANGLE-SUM PROPERTY OF A TRIANGLE

$T + R + U = 540^\circ$ or six right angles; $P + Q + R$ is the difference, namely 180° or two right angles.

Any angle, such as S or T or U , between one side of a triangle and another side produced is called an exterior angle of the triangle. What is the relation between S, R, Q ? Between T, R, P ? Between U, P, Q ?— S together with P makes 180° , and $Q + R$ together with P make 180° , so that $S = Q + R$.

The pupils will now be able at once, or with a little questioning, to state the results formally:—

The 3 angles of a triangle together make 180 degrees or 2 right angles.

Any exterior angle of a triangle is equal to the two interior and opposite angles taken together (that is, to the two angles of the triangle that are not beside the exterior angle).

Discuss the results you arrive at by setting out in the other direction round the triangle of Fig. 5.—The results are as before, except that the angles vertically opposite S, T, U take the place of S, T, U .

Repeat the reasoning by which you saw that vertically opposite angles are equal (chap. II, art. 13).

8. [*Note for the teacher.* The conclusion that in walking round a room, or round a triangle, one turns through 360° differs from the preceding experimental conclusions. It is not a necessary result of the fact that in turning round on one spot one turns through 360° ; in fact, it is possible to suppose that the angle turned through in going round a room or a triangle differs from 360° and to work out a geometry on that supposition. It is only a conclusion that is in keeping with all our experience; it is what we may call an intuition. A geometry worked out on this intuition represents our experience very well, while the geometries worked out on other suppositions fail in some respects to represent our experience.

This experimental or experiential result may be made the foundation of the theory of parallels, in place of the experimental result of chap. II, art. 2. To this we now proceed.

These remarks are of course unsuitable for consideration by children; they are for the teacher.]

9. Draw a straight line, and at two points A and B on it erect perpendiculars AC and BD (Fig. 6). At a number of points on BD erect perpendiculars EF , &c., to BD . Measure the angles these perpendiculars EF , &c., make with AC and the length of each perpendicular intercepted between BD and AC . We have thus the experimental result that any perpendicular to BD is also perpendicular to AC , and that the perpendicular distance between BD and AC is the same all along.

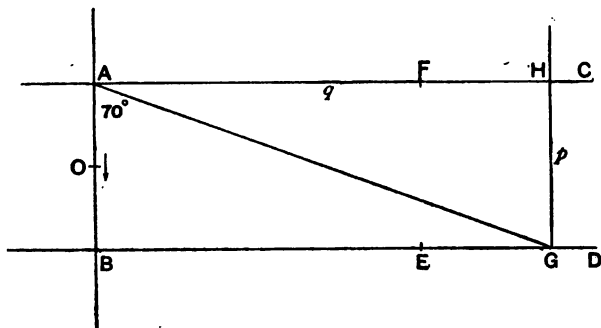


FIG. 6.

We shall now see that these properties follow from the fact that the three angles of a triangle together make two right angles, which we will for shortness call the angle-sum property of a triangle.

10. In Fig. 6 draw AG making an angle of 70° with AB . At G , where this line meets BD , draw GH perpendicular to BD , meeting AC in H . Now, without measuring, find the value of all the other angles of the figure.—Since GAB and GAB make 90° , GAH is 20° . The angle-sum property applied to the triangle ABG , in which $\angle A = 70^\circ$ and $\angle B = 90^\circ$, gives $\angle BGA = 20^\circ$. Then since BGA and AGH make 90° , AGH is 70° . The angle-sum property of the triangle AGH now gives $\angle AHG = 90^\circ$. And so on.

Check these values by measuring all the angles.

Are there two congruent triangles in the figure, that is, two triangles such that if one was cut out it could be fitted on to the other?—If $\triangle AGH$ is cut out and turned round

128 THE ANGLE-SUM PROPERTY OF A TRIANGLE

so that its angle of 70° comes to A and its angle of 20° to G (which can be done, since the distance between these angles is AG); then its side p will lie along AB and its side q along GB , and the triangles will fit. So that the distance GH between AC and BD is equal to the distance AB between these lines.

Actually cut out the triangle AHG of your figure and fit it on ABG .

11. Make Fig. 6 over again, but this time draw the line AG making any angle with AB , and suppose that this angle contains x degrees. Express all the other angles of the figure in terms of x , and repeat Art. 10 with the necessary changes. We find that in this case also GH is at right angles to AC and is equal to AB . And in this case the point G may lie anywhere on BD , so that we know that any line perpendicular to BD is also perpendicular to AC , and that the perpendicular distance between AC and BD is the same all along.

GH was drawn perpendicular to BD . Could you show the same results by beginning with a line drawn perpendicular to AC ?

The lines AC and BD are said to be parallel. In general two lines are called parallel when a perpendicular to one is perpendicular also to the other and the distance between them measured along the perpendicular is the same everywhere.

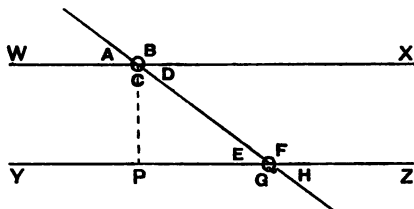


FIG. 7.

12. Draw two parallel lines (by making both perpendicular to a third), and draw another cutting them in O and Q (Fig. 7). Measure all the 8 angles of the figure. How many different sizes of angles are there?

Repeat the figure, measure no angles, but name them as in Fig. 7, and express all the other angles in terms of A .—We draw OP perpendicular to one of the parallels, and therefore perpendicular to the other; use the angle-sum property for $\triangle OPQ$; and find that D , E , H are equal to A , while B , C , F , G are equal to $180 - A$ degrees.

When a line falls across two other lines, as in Fig. 7, certain names are given to pairs of angles, thus:

D and E	are called alternate angles;
C and F	" " "
A and E	are called corresponding angles;
B and F	" " "
C and G	" " "
D and H	" " "

Express your results by means of these names.—A line falling across two parallel lines makes alternate angles equal to one another and makes corresponding angles equal to one another.

The line falling across the others is often called a transversal.

13. If a line falls across two other lines so as to make a pair of alternate angles equal, can you tell whether the two lines are parallel? Or if the line falls so as to make a pair of corresponding angles equal? First take a line OQ and draw two other lines so that two alternate angles, say D and E , have the value 40° ; draw a number of perpendiculars from one line to the other, test if they are perpendicular to the second line, and if all these perpendiculars are of equal length. Do the same making two corresponding angles, say B and F , equal.

Then try to settle the matter by general reasoning.—Suppose OQ falls across WX and YZ so as to make $D = E$. Draw OP perpendicular to WX (Fig. 7). Then the angle-sum property shows that $E + \angle POQ + \angle OPQ = 180^\circ$; and E being equal to D , $E + \angle POQ$ is equal to $D + \angle POQ$, that is, to 90° ; so that $\angle OPQ$ is equal to 90° . We know then that WX and YZ are both at right angles to OP , so that any perpendicular to one of them is perpendicular to the other, and the perpendicular distance between them is the same all along. That is, WX and YZ are parallel.

180 THE ANGLE-SUM PROPERTY OF A TRIANGLE

The case of OQ falling on WX and YZ so as to make the other pair of alternate angles (C and F) equal, and all the cases of corresponding angles are left as exercises.

The conclusion is that if a line falls across two others so as to make a pair of alternate angles, or a pair of corresponding angles equal, the two lines are parallel. In this case, as always, the pupils should be led to give the statement of the proposition.

Constructions for parallels.

14. Are any of your results of use for drawing parallels in a simpler way than by making the two parallels perpendicular to the same line?—We have already (Art. 13) drawn parallels by making a pair of alternate angles equal, or a pair of corresponding angles equal. Does this method enable you to draw through a given point C a line parallel to a given line AB (Fig. 8)?—Draw any

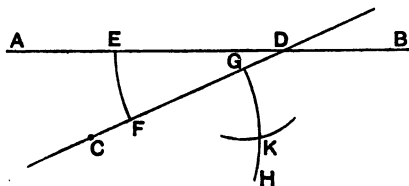


FIG. 8.

line through C to meet AB in D . With D as a centre draw an arc EF of a circle from DA to DC ; and with centre C and the same radius draw an arc GH , G lying on CD . Open your compasses to reach from E to F , and with centre G and this radius draw an arc cutting GH in K . Join CK . The alternate angles ADC and DCK which the transversal CD makes with AB and CK being equal, CK is parallel to AB . Repeat the proof that the angles ADC and DCK are equal.

Ex. 1. Draw a parallel to a given line, using only a piece of paper or a piece of tracing-paper to measure angles and to carry angles.

15. Various mechanical means are used to draw parallels. In each case the pupils should be provided with the apparatus and left to discover how to use it.

One means is a wooden triangle called a set-square. One edge p of the set-square is placed against the given line AB (Fig. 9). A ruler is placed against a second edge q of

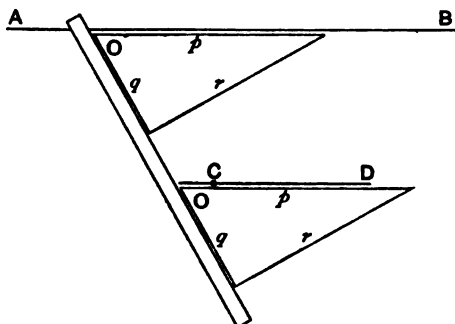


FIG. 9.

the set-square, and the set-square is slid along the ruler till the edge p comes against the point C through which the parallel is to be drawn. Show that the line CD ruled along the edge p of the set-square is parallel to AB .—The edge of the ruler is in this case the transversal, and the two lines AB and CD make with it equal corresponding angles, each angle being the angle O of the set-square.

Another means is the drawing-board and T-square (Fig. 10). The T-square is a ruler AB with a cross-piece CD screwed on below one end, so that when the ruler lies on the board the cross-piece is below the surface of the board, and fits against the side of the board. Show that all the lines ruled with CD in contact with the same side of the board are parallel.

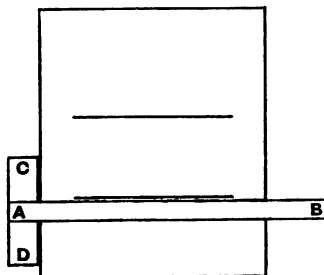


FIG. 10.

If a set-square is set on the T-square and a line ruled against an edge of the set-square, and the T-square slid

182 THE ANGLE-SUM PROPERTY OF A TRIANGLE

along the board while the set-square slides along the T-square, will a second line ruled against the same edge of the set-square be parallel to the first line?

16. A third means is a pair of parallel rulers (Fig. 11). A and C are two points on one ruler so placed that AC is

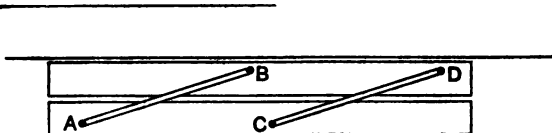


FIG. 11.

parallel to the edges, B and D are two points on the other such that BD is parallel to its edges, and the distance BD is equal to the distance AC . Two equal rods AB and CD are hinged to these points. Now hold one ruler still, move the other about and rule a number of lines against it. Can you show that these are all parallel?

Measure AC or BD and one of the rods, and draw the four-sided figure (or quadrilateral) $ACDB$ in a number of its possible shapes, keeping AC in the same position for them all. What do you observe about the various positions of BD ?

Fig. 12 being one of the possible positions, and the lengths AB and AC being of course known, what further

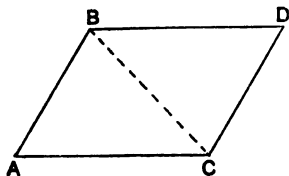


FIG. 12.

measurements do you need to make to be able to copy the figure?—The length BC would do, or the angle BAC , or the length AD , one measurement more in every case. Suppose BC measured and the figure drawn, including BC . Do you see in Fig. 12 any pair of congruent triangles, that is,

any pair of triangles that would fit on to one another?—The triangles ABC and DBC have $AB = CD$, $AC = BD$, and $BC = BC$, so that they are congruent. Repeat the proof by which it was shown that the two triangles can be made to coincide, and name the equal pairs of

angles.— $\angle ABC = \angle BCD$, $\angle ACB = \angle CBD$, $\angle BAC = \angle BDC$.

What follows as to the parallelism of lines of the figure?—The transversal BC makes equal alternate angles with AC and BD , and also with AB and CD , so that AC is parallel to BD and AB to CD .

The symbol $\parallel$ is used as a contraction for parallel or for is parallel to, so that the results may be written

$$AC \parallel BD,$$

$$AB \parallel CD.$$

17. Can you now justify the use of these parallel rulers? Can you show that two positions of the edge of the ruler are parallel?—Suppose p and r in Fig. 13

to be two positions of BD , and q and s the corresponding positions of the edge of the ruler. Draw any transversal t . Then from the way B and D were taken on the ruler (Fig. 11) p is parallel to q , and r to s , so that the transversal t makes

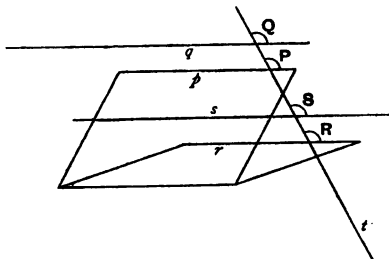


FIG. 13.

equal corresponding angles P and Q with p and q , and equal corresponding angles R and S with r and s . Also p and r are parallel so that $\angle P = \angle R$. Thus we have $\angle Q = \angle P = \angle R = \angle S$, and the corresponding angles Q and S being equal the lines q and s are parallel.

18. We found that the figure $ABDC$ of the parallel rulers had each pair of opposite sides parallel. A 4-sided figure with each pair of opposite sides parallel is called a parallelogram. That $ABCD$ is a parallelogram was a consequence of having $AB = CD$ and $AC = BD$. State a converse proposition, that is, one in which the data and the consequence are interchanged; and see if this new proposition is true.—A converse is that if each pair of opposite sides of a 4-sided figure are parallel, each pair of opposite sides are also equal, or a parallelogram has each pair of opposite sides equal. Draw a few

184 THE ANGLE-SUM PROPERTY OF A TRIANGLE

parallelograms, and measure the opposite sides. Then discuss whether the proposition is true.—It is true, for if $BD \parallel AC$ and $BA \parallel CD$ (Fig. 12), drawing the transversal BC gives two triangles ABC and BCD in which $\angle ABC = \angle DCB$, $\angle ACB = \angle DBC$, and $BC = BC$, that is, two congruent triangles in which $AB = CD$ and $AC = BD$.

19. See whether a 4-sided figure $ABDC$, in which BD is equal to AC and also parallel to AC , is a parallelogram. First, draw a few figures from these data.—This time the 2 triangles (Fig. 12) have $BD = AC$, $BC = BC$, $\angle DBC = \angle BCA$, so that they are congruent and $\angle ABC = \angle BCD$; which shows AB to be parallel to CD .

See whether a 4-sided figure $ABDC$ in which $BD = AC$ and $AB \parallel CD$ is a parallelogram. First, draw a few figures from these data.—We lay down BD and draw BP and DQ parallel to one another. Then we take C anywhere on DQ and with C as centre and a radius equal to BD we draw a

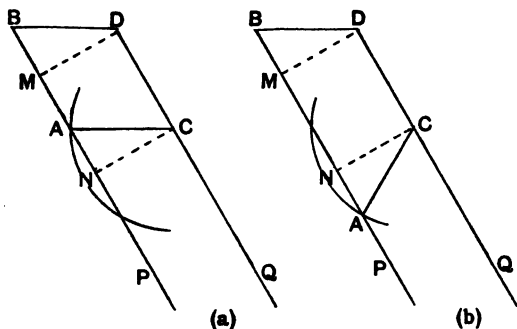


FIG. 14.

circle cutting BP in A (Fig. 14). This Figure $BDCA$ has $BD = AC$ and $AB \parallel CD$.

Is the figure a parallelogram, that is, is AC parallel to BD ?—Two positions are possible for A , namely the two intersections of the circle with BP . In one of the figures (Fig. a) AC is parallel to BD as nearly as we can measure; in the other (Fig. b) AC is obviously not parallel to BD .

20. Draw DM and CN perpendicular to DC meeting BA in M and N . What more do you know of DM and CN ?—That DM and CN are both $\perp$ to AB ($\perp$ is shorthand for 'perpendicular' or 'is perpendicular to'), and that $DM = CN$, each being the distance between the parallels. Are there any congruent triangles in the figure?—Triangles BDM and CAN are congruent; for they have the angles at M and N equal, the sides DM and CN equal, and the sides DB and CA equal; and the sides opposite the equal angles are greater than the other pair of equal sides.

How do you know that BD is greater than DM ?—We saw that, given the lines BD and BP , the least radius of a circle with centre D that would meet BP was the perpendicular from D on BP .

Repeat the proof that two triangles can be fitted on to one another if they have two pairs of sides equal and the pair of angles opposite the greater sides also equal (chap. V).

Can you now tell whether either of the cases of Fig. 14 is a parallelogram?—From the congruence of DBM and CAN we know that the angles DBM and CAN are equal, or, in other words, BD and CA are equally inclined to BA . In the case in which the equal angles are corresponding angles made by the transversal AB with the lines BD and AC (Fig. 14a) the figure $BDCA$ is a parallelogram. In Fig. 14b the equal angles are not corresponding angles, and the figure $BDCA$ is not a parallelogram.

Show that in Fig. 14b BD and AC are equally inclined to CD also.

21. Let us now summarize our results. One small point not explicitly proved is left as an exercise.

The diagonal of a parallelogram (that is, the line joining two opposite corners) divides the parallelogram into two congruent triangles; each pair of opposite sides are equal; each pair of opposite angles are equal.

Also a four-sided figure is a parallelogram and has these properties, provided each pair of opposite sides are equal; or provided one pair of opposite sides are both equal and parallel.

If a four-sided figure has one pair of opposite sides equal and the other pair parallel, the equal

186 THE ANGLE-SUM PROPERTY OF A TRIANGLE

pair are equally inclined to the parallel pair, but two kinds of figure are possible, only one of which is a parallelogram.

Ex. 2. A four-sided figure or quadrilateral having the four sides a b c d , and the four angles P Q R S , as in Fig. 15; find whether it is a parallelogram :—

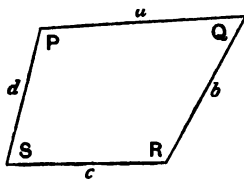


Fig. 15.

- if $a \parallel c$ and $P + Q = 180^\circ$,
- if $a \parallel c$ and $P + S = 180^\circ$,
- if $a \parallel c$ and $P = R$,
- if $a = c$ and $P + Q = 180^\circ$,
- if $a = c$ and $P + S = 180^\circ$,
- if $a = c$ and $P = R$,
- if $P = R$ and $S = Q$,
- if $P = R$ and $P + S = 180^\circ$.

22. Draw two lines AB and AD at right angles. Through B draw a line parallel to AD and through D a line parallel to AB , these lines meeting in C . What can you tell of the angles of this figure by general reasoning? And if a number of lines perpendicular to AB and CD and ending on these lines were drawn? Such a parallelogram, with its angles all right angles, is called a rectangle. We saw a good deal of rectangles in our discussion of area, and we used the terms length and breadth without inquiry into their exact meaning. We now see that the length is the perpendicular distance between one pair of parallel sides, which distance is the same all along, and the breadth the distance between the other two sides.

What name do you give the rectangle when its sides are all equal in length?—It is a square.

Name a set of data that will determine a rectangle?—The length and breadth will do. Name a set that will determine a square.—The length of the side is enough.

23. Areas.—In chapter III we found experimentally that the diagonal of a rectangle divides it into two parts of equal area. Can you now give a formal proof?—It has been proved for a parallelogram, and a rectangle is a particular kind of parallelogram. Repeat the proof for the case of the rectangle.

We found further, experimentally (chap. III, art. 8), that

if the acute angles of a right-angled triangle are folded against the right angle, the result is a rectangle exactly covered by two triangles. Can you justify this result?

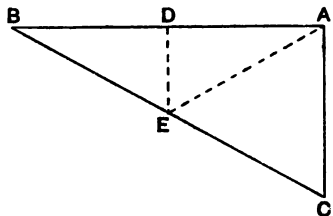


FIG. 16.

First suppose B folded against A (Fig. 16), and mark all the angles of the figure that you know, or can express in terms of B .—

$$\angle EDB = 90^\circ.$$

$$\angle EDA = 90^\circ.$$

$$\angle EAD = \angle B, \text{ since they fit together.}$$

$$\angle EAC = 90^\circ - \angle EAD = 90^\circ - B.$$

$\angle ECA = 90^\circ - B$, since $\angle A B C$ together make 180° and $A = 90^\circ$.

What can you conclude as to the lengths EB , EA , EC ?—They are all equal; $\angle EBA = \angle EAB$ and $\angle ECA = \angle EAC$, so that we have two isosceles triangles. And if we

fold C on A ?—The crease bisects CA at right angles (say at F) and (since E is equidistant from C and A) passes through E .

What is the size of the angle DEF ?—It is 90° , since the other angles of the quadrilateral $ADEF$ are right angles.

Thus it is proved that folding B and C against A gives a rectangle $ADEF$ exactly covered.

Ex. 3. Show that a circle with centre at the middle of the hypotenuse of a right-angled triangle can be drawn to pass through the 3 vertices; and that if a circle is drawn, and B and C are at the two ends of a diameter, and A anywhere on the circumference, the angle BAC is a right angle.

Use this property to give a construction for a right angle.

Ex. 4. In article 3 of the present chapter a triangle of any shape was folded experimentally into a rectangle of two thicknesses of paper. Prove this result by general reasoning.

138 THE ANGLE-SUM PROPERTY OF A TRIANGLE

Ex. 5. In chap. III, art. 12, a triangle was made from a rectangle by cutting off two corners, and these corners were experimentally fitted to the triangle to exactly cover it. Prove this result by general reasoning, and thus justify the expression found for the area of a triangle, half the product of the base by the height.

24. Cut out a paper parallelogram $ABCD$ (Fig. 17) and

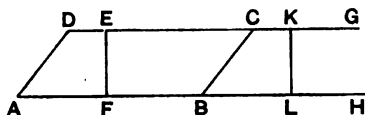


FIG. 17.

see if you can cut it in two and fit it together into a rectangle.—Cut it along a line EF perpendicular to two opposite sides, and then fit BC and AD together.

You thus have the experimental result. Can you prove it to be more than experimental, to be more accurate than errors of execution and observation make it possible for a mere experiment to show?—Produce DC to G and AB to H . AD and BC are equal and so can be fitted together. The corresponding angles ADE and BCG are equal, so that DE will lie along CG . The angle DAF is equal to the corresponding angle CBH , so that AF will lie along BH . Thus $ADEF$ will lie as $BCKL$, and the angles made by KL with CK and BL are the angles DEF and AFE , that is, right angles. Therefore the fitting together makes a rectangle $EFLK$.

Give an expression for the area of the parallelogram.—It is the same as of the rectangle, namely $FL \times EF$, or, since

FL is simply the length AB , it is $AB \times EF$, the product of a side by the distance between it and the parallel side. Or, using the usual terms, it is the product of the base by the height.

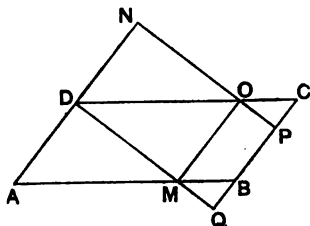


FIG. 18.

Does it matter which side you choose for base?—If in the case of Fig. 17 we chose AD for base, we could not draw a perpendicular to AD that would meet BC . DM (Fig. 18) is the best we could do. Make the parallelo-

gram in paper, cut off the piece ADM and fit it in the position DNO . What will you then do to make the figure a rectangle?—Cut off the piece OCP and fit it in the position MBQ .

Prove that this gives a rectangle, and therefore, in this case also, the area is the product of the base and the height.

Ex. 6. Cut out a long thin parallelogram that will have to be cut into several pieces to fit again into a rectangle, and show how to do it.

Ex. 7. Cut out a piece of paper of the form called a trapezium (Fig. 19), that is, a four-sided figure with one pair of opposite sides

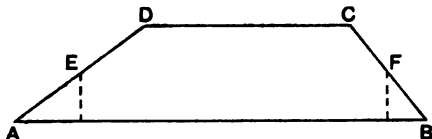


FIG. 19.

parallel, and show how to cut it up and make it into a rectangle. (Take E and F the middle points of AD and BC , and cut along EF from these points to AB .)

Also give an expression for the area of a trapezium in terms of the lengths of the parallel sides and the distance between them. Calculate the area of Fig. 19 from measurements.

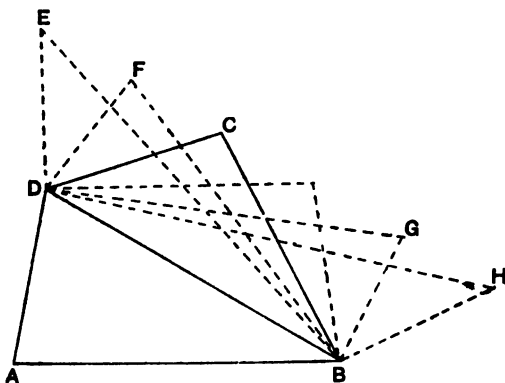


FIG. 20.

25. Draw any quadrilateral $ABCD$, and find its area.—

140 THE ANGLE-SUM PROPERTY OF A TRIANGLE

We join BD , draw perpendiculars from C and A on BD , and use for each triangle the expression 'half the base by the height'.

Draw a number of triangles on BD as base and equal to CBD in area, as EBD , FDB , &c. (Fig. 20). What do you notice about these triangles, or what property did you use to draw them?—They all have the same height, so that $EFCGH$ is a straight line parallel to BD .

Can you make a single triangle equal to the quadrilateral $ABCD$?—If BH is in line with AB , the two triangles ABD and HBD form a single triangle AHD . To make DF in line with AD would also do.

State formally how to make a triangle equal in area to a quadrilateral $ABCD$.—Through C draw a parallel to BD to meet AB produced in H . Then AHD is a triangle equal to $ABCD$.

26. In the same way make a quadrilateral equal in area to a given five-sided figure. Show how to make a triangle equal to any polygon, that is, to any area bounded by straight lines.

27. Draw a triangle with sides of 14, 11, 7 cm., and try to make a square equal to it. If the side of the square is x cm., what do you know of x ?—We measure the height of the triangle and so find the area to be 37.9 sq. cm. Then $x \times x = 37.9$, and our graph of $x \times x$ (chap. III, art. 17) gives us $x = 6.2$, and we draw the square.

This method is a trifle tedious and not very exact; we may find a better method later.

Show how to make a square equal to any polygon.

EXERCISES.

8. Bisect the angles of a rectangular sheet of paper by folding adjacent sides together, thus obtaining four crease lines running across the paper from corners to sides. At the middle of the paper a quadrilateral is thus formed by the creases. Determine its shape in any way you like, and justify your statement by a geometrical proof.

9. Use the angle-sum property to make a triangle having $a = 11$ cm., $A = 58^\circ$, $C = 65^\circ$, by calculating B and then using the values of a , B , C .

Draw any two acute angles and mark them A and C . Without measuring A and C , make a triangle having $a = 11$ cm. and A and C

equal to the angles you have drawn. (Find B by laying down an angle of 180° and cutting from it the angles A and C .)

10. Draw a figure $ABCDE$ having five equal sides and five equal angles. First calculate the value in degrees of an angle.

If the sides of the figure were freely jointed at A, B, C, D, E , and if BC and AE were turned about B and A respectively until each was at right angles to AB , what would then be the values of the angles at C, D , and E ?

11. ABC (Fig. 21) represents a triangle drawn on tracing paper, which is placed over a fixed triangle DEF so that AC is parallel to EF . Sketch a large figure (which need not be to scale), calculate the angles K, L, M, N , and write in each its value in degrees. The method of obtaining the results should be briefly indicated, and an angle referred to by one letter only.

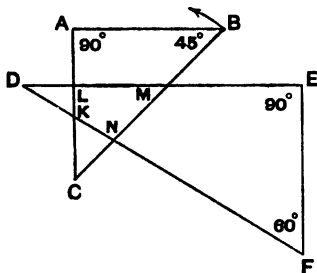


FIG. 21.

If a pin was pushed in at C and the tracing-paper revolved, as shown by the arrow, through an angle of 10° , what would then be the values of the angles K, L, M, N ?

12. Make three triangles, calling each ABC . In each bisect the angle A by a line AM , and from B draw BM perpendicular to AM . In each figure measure $\angle B, \angle C, \angle MBC$. Make a table like that below, and fill it up.

—	$\angle B$	$\angle C$	Difference of $\angle B$ and $\angle C$	$\angle MBC$
Fig. 1				
Fig. 2				
Fig. 3				

What inference can you draw from a comparison of the last two columns? See if you can justify your inference by geometrical reasoning.

13. $ABCD$ is a quadrilateral figure in which AB is parallel to CD but not equal to it. Draw a figure of this kind, making $AB=10$ in., $BC=4$ in., $CD=5.2$ in., $BD=7.1$ in., and fold the page so that AB and CD fall together. Measure the part of the

142 THE ANGLE-SUM PROPERTY OF A TRIANGLE

crease between AD and BC and compare it with half the sum of AB and OD .

Now draw perpendiculars from O and D to AB and from A and B to the line of the crease produced, and give a geometrical proof of the relation between the lengths of AB , OD , and the crease.

14. From a corner A of a rectangular sheet of paper measure off AB along one edge = 24 cm., and AO along the other = 19 cm. Join BO and cut along this line. Fold the triangle to make a crease AO through A and at right angles to BO , meeting BO in O . Open the triangle out and fold the corners A , B , O over to meet at O . Open the triangle out again and, in any way you please, make a full-size drawing of it, showing the creases by dotted lines. Then answer either (a) or (b).

(a) State whether the three corners meet so that the part folded down just covers up the rest of the triangle, giving reasons why this should or should not be the case.

(b) What conclusion may be drawn about the sum of the angles of the triangle, and about the area of the triangle? Give an independent proof of one of these conclusions.

15. Imagine a triangle cut out and placed on the desk before you in any definite position. State precisely in each case the measurements you would take (a) to draw a figure of the same shape; (b) to draw a figure of the same size and shape; (c) to enable you to make a drawing of the figure in exactly the same position if the original were removed. You are to give the minimum number of measurements necessary in each case.

16. The four bars of a bicycle frame taken in order have lengths 23, 58, 62, 56 cm. Take four strips of cardboard and join them by eyelets or paper fasteners into such a frame, making each side $\frac{1}{4}$ th of its true length; the joints to be loose.

Is the form of the frame settled by the given lengths? How would you add another bar to fix the form?

An actual bicycle frame is fixed in shape with no such additional bar. How is that?

17. A four-sided field has two sides parallel and of lengths 260 and 200 yards, a third side $\perp$ to them and 220 yards long. The fourth side is bounded by a footpath which is to be moved so as to make the field rectangular without loss or gain of area. Draw the field, using 1 cm. to represent 20 yards, and show how the fourth side must be altered.

What is the area of the field (in acres)?

18. $ABOD$ is a rectilinear figure having AD parallel and unequal to BC . Show, with proof, how to make a rectangle equal to $ABOD$.

AA' , BB' , CC' , DD' , EE' are parallel straight lines. The distance between each and the next being h ft., and their lengths a , b , c , d , e ft. respectively, find an expression for the area of the figure $A'B'C'D'EEDCBA$. When $a = 2.7$, $b = 7.3$, $c = 8.2$, $d = 6.3$, $e = 2.4$, $h = 3.1$, draw the figure to scale, calculate the area, and mark it on the figure. Is the figure completely specified?

19. A 20-lb. steel plate originally measures 10 ft., by 4 ft. 6 in. at one end and 4 ft. 9 in. at the other, the width measurements being square to the 10 ft. edge. It is first sheared parallel as far as possible to a width of 4 ft. 7 in., next one square corner measuring 3 ft. $\times$ 2 ft. is cut off, next a circular hole 12 in. in diameter is cut in it, and, finally, a rectangular manhole measuring 23 in. by 15 in. is cut in it.

What is the weight in lb. of the remaining piece of the original plate?

20. Write down the values of $1 \div x$ for the following values of x : 0.01, 0.1, 0.5, 1, 2, 5, 10. Show by a graph how $1 \div x$ varies as x increases from 0.1 to 10, taking one inch as unit.

From the graph read off the value of $1 \div 2.7$. Why is the graph not very useful for reading off the value of $1 \div 0.27$?

21. Water has to be diverted from a river through a six-inch diameter pipe running full bore at a velocity of one foot per second to irrigate a field of 20 acres. How long will it take to deliver an inch of water over the whole area?

22. Suppose you are asked to make a copy of a four-sided figure $ABCD$ in which AB and CD are parallel, the figure being drawn on a sheet of paper of the size that you are using. How many measurements would you have to make to reproduce the figure (1) when your copy is to have the same position on your sheet that the given figure has on its sheet; (2) when the position on the sheet does not matter? Give, in each case, one set of measurements which would be just enough, and state briefly how you would use these to draw the figure.

23. A pint of milk weighs a pounds, a pint of water weighs b pounds, and a pint of a mixture of the two weighs c pounds. What fraction (by volume) of the mixture is milk?

If $b = 1.125$, $a = 1.128$, and $c = 1.127$, what fraction is milk?

CHAPTER IX

ON DRAWING TO SCALE

1. In the course of our discussions we have frequently used figures showing objects not full size, but drawn to some smaller scale. For instance, in chapter V we had a map of a piece of country, and in chapter VII we had models of cranes smaller than the real cranes, and drawings of cranes still smaller. What was the relation between lengths on the map, or the model, or the drawing, and the corresponding lengths on the original object?—In each case all the lengths on the copy were the same fraction of the original lengths.

And what was the relation between the angles of the copy and the angles of the original?—Each angle of a copy was equal to the corresponding angle of the original object. This was assumed to be so, without any special statement. We will now discuss the truth of the assumption.

2. By the help of a measuring-tape only make a plan of the playground or of a field, making every length on the plan $\frac{1}{500}$ of the actual length, that is, using a scale on

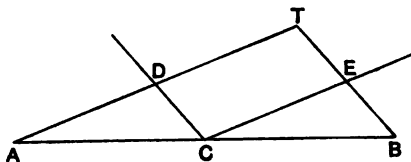


FIG. 1.

which 2 mm. represent a metre.—We begin with (say) a straight piece of fence AB , measure it and lay down the corresponding length. [Figure 1 and the following figures in the text are mere sketches, and not drawn to any

particular scale.] Then we measure the distances of the tree T from A and B . Each distance gives a circle on which T lies, and their intersection gives T .

If your tape will not reach from A to T , how do you make sure of measuring in a straight line from A to T ?—One boy stands at A with one end of the tape and looks towards the tree. A second boy carries the other end towards the tree, keeping in the first boy's line of sight, and so in a straight line to the tree. When the tape is stretched, the first boy comes up, and the second goes on as before towards the tree.

You rule AT on your plan with a ruler. How could you test whether the edge of the ruler is straight?—By looking along it. Or by ruling a line with it, and laying it in all possible ways against the line to see if it fits.

8. Draw another plan on a scale of 1 mm. to 1 metre (Fig. 2, sketch). How do the lengths $A'B'$, $A'T'$, $B'T'$ of this figure compare with the corresponding lengths of Fig. 1? Compare the angles by measuring.

In your Fig. 1 bisect AB in C and draw CD and CE parallel to BT and AT . What equalities of lengths, angles, and areas can you now show about Fig. 1 by general reasoning?—Since DC and BT are parallel, $\angle DCA = \angle EBC$; since AT and CE are parallel, $\angle DAC = \angle ECB$; and we made $AC = CB$. So that the triangles ADC and CEB are congruent, that is, will fit exactly if placed one on the other (chapter V). Since $DTEC$ is a parallelogram, $DT = CE$ and $CD = ET$ (chapter VIII). From the congruence of $\triangle s$ ACD and CBE we know that $CE = AD$ and $CD = EB$. So that $AD = DT$, and each is half of AT ; and $BE = ET$, and each is half of BT .

Can you show $A'B'T'$ congruent to any triangle in Fig. 1?— AB is $1/500$ of the length of the fence and $A'B'$ is $1/1000$ of the same length; AT is $1/500$ and $A'T'$ $1/1000$ of the distance of the tree from the point A of the fence; BT is $1/500$ and $B'T'$ $1/1000$ of the distance of the tree from B . So that $A'B'$, $A'T'$, and $B'T'$ are the

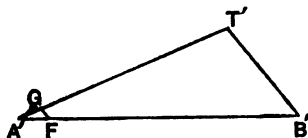


FIG. 2.

halves of AB , AT , and BT ; that is, they are equal to AC , AD , and CD , and also equal to CB , CE , and EB . We know, therefore, that $\triangle A'T'B'$ is congruent to $\triangle ADC$ and to $\triangle CEB$ (chapter V); and this gives what we wanted to know, that the angles at A and A' are equal, that the angles at B and B' are equal, and that the angles at T and T' are equal since each is equal to the angle ADC .

4. Ex. 1. How does your proof tell you that the angle at A' is equal to that at A , and not to the angle ADC or the angle ACB ?

Ex. 2. Certain reasoning was quoted to show $CE = DT$, to show $\triangle s ACD$ and CBE congruent, and to show $\triangle s A'B'T'$ and AOD congruent. Give the reasoning in full.

Ex. 3. Show that $\triangle ATB$ can be cut up into four triangles, each of which will exactly fit $\triangle A'T'B'$.

5. Now make a plan of the fence and tree on a scale of 3 mm. to 1 metre. Call the ends of the fence on the plan X and Y , and the tree Z (Fig. 3). Divide XY into three

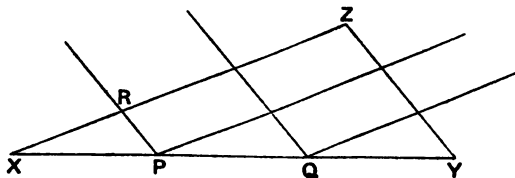


FIG. 3.

equal parts by cutting off XP and PQ equal to $A'B'$. Through P and Q draw parallels to XZ and YZ , and by the same kind of reasoning as before show the three angles of $\triangle XYZ$ to be equal to the three angles of $\triangle A'B'T'$ and to the three angles of $\triangle ABT$.

6. Ex. 4. Compare the areas of the triangles ABT , $A'B'T'$, and XYZ . Compare them also with the area of the triangle formed by the tree and the two ends A and B of the actual fence.

Ex. 5. Justify the following method of trisecting a straight line PQ , that is, of dividing it into three equal parts:—From P draw any line PX . Take any length on your dividers and mark off PA , AB , BC of this length along PX . Join C to Q , and

through A and B draw parallels to OQ . These parallels trisect PQ .

Ex. 6. Give some other way of trisecting the line PQ .—Measure PQ , divide the length by 3 arithmetically, and measure this distance off along PQ .

7. Show that the angles of $\triangle ATB$ are equal to those of the triangle formed in the playground by the tree and the two ends of the fence.—Suppose the length AB laid off along the fence as often as possible. Laid off 500 times it will just cover the fence. From all the points of division suppose parallels drawn, just as in Fig. 8. Then we shall have 500 triangles all congruent to $\triangle ATB$, and a great many parallelograms, and the proof runs as before.

Show that the angles of $A'T'B'$ and XYZ are equal to those of the triangle made by the tree and the fence.—The proof as to $A'T'B'$ does not differ from that for ATB . In the case of $\triangle XYZ$ if we lay off XY repeatedly along the fence, 333 lengths will not cover the fence and 334 are too long. Let us divide up $\triangle XYZ$, as in Fig. 8, and use $\triangle XPR$. XP can be laid off an exact number of times along the fence and an exact number of times along XY . And the former proof holds as between $\triangle XPR$ and the triangle on the playground, and also as between $\triangle XPR$ and $\triangle XYZ$.

8. Suppose LMN (Fig. 4) a plan drawn to a scale of 1·3 mm. to 1 metre, L corresponding to A , M to B , and N to T . Show that the angles of LMN are equal to those of $A'B'T'$.—We want a length that can be laid off an exact number of times along LM and also along $A'B'$. Such a length is $A'F$, $1/10$ of $A'B'$, which is equal to LK , $1/18$ of LM ; for each of these represents the fence on a scale of 0·1 mm. to 1 metre. Draw $FG \parallel$ to $B'T'$, and $KU \parallel$ to MN . Then we show, as before, that $A'G$ is $1/10$ of $A'T'$, FG is $1/10$ of $B'T'$, LU is $1/18$ of LN , and KU is $1/18$ of MN ; that $\triangle A'FG$ and $A'B'T'$ have their angles equal, and that $\triangle LKU$ and LMN have their angles equal.

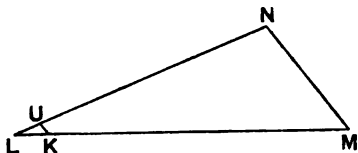


FIG. 4.

Also each of the Δs $A'FG$ and LKU represents the fence and tree on a scale of 0.1 mm. to 1 metre, so that these two triangles have their angles equal. Therefore Δs LMN and $A'B'T'$ also have their angles equal.

Ex. 7. How would you divide $A'B$ into 10 equal parts?—We could measure $A'B$ (we really know this length already; we used it to make the plan $A'B'T'$), divide the length by 10 arithmetically, and lay this distance off repeatedly along $A'B$. Or we could draw any line through A' , lay off any convenient distance along it 10 times, join the 10th division to B and draw parallels through the other 9 points of division.

How would you divide LM into 13 equal parts?

Ex. 8. To compare the Δs $A'B'T'$ and LMN , instead of marking off $1/10$ of $A'B$ and $1/13$ of LM , we could have produced $A'B'$ to 13 times its length, and produced LM to 10 times its length. Complete the comparison of the triangles in this way.

9. Now consider another method of making a plan, in which we measure only one length, say the fence, and for the rest measure angles only, so that the position of the tree is fixed by the angles TAB and TBA . In what cases is the advantage of this method greatest over the method of measuring only lengths by a measuring-tape or a surveyor's chain?—The advantage is greatest when the area considered is large. Then restriction to the tape or chain would mean walking many kilometres. On the present plan all the measurements can be taken from a few points.

10. Let us by this method make plans of the fence and tree to the same scales as before, scales on which 1 metre is represented by 2, 1, 3, 1.8 mm. in succession. In the first case we draw AB $1/500$ of the length of the fence, on a sheet of paper pinned to a drawing-board. Then we set up our drawing-board (or plane-table) at A in a horizontal position, and turn it round till the line AB points along the fence. We then draw a line from A in the direction of the tree; the point T that represents the tree will lie somewhere on this line. Now we transfer our drawing-board to the other end of the fence, make BA point along the fence, and draw from B a line towards the tree. The point T lies where this line crosses the former one (Fig. 1).

The other cases differ only in having $A'B'$ (Fig. 2) $1/1000$ of the length of the fence, XY (Fig. 3) $3/1000$ of

the length of the fence, and LM (Fig. 4) $1\cdot8/1000$ of the length of the fence.

11. Having drawn Fig. 1 in this way, can you show that AT represents the distance between the end of the fence and the tree on the scale adopted? That is, can you show that AT is $1/500$ of that distance? And $A'B'$ being half of AB , can you show that $A'T'$ is also half of AT ? LM being $1\cdot8$ times as long as $A'B'$, can you show that LN is $1\cdot8$ times $A'T'$? And LM being $1\cdot8/1000$ of the length of the fence, can you show that LN is $1\cdot8/1000$ of the distance of the tree from the end of the fence?

These proofs follow closely the lines of the former ones for the plans made by the measuring-tape, that is made by the method of 'chain surveying'. Thus, if in Fig. 1 C is the middle point of AB , $CD \parallel$ to BT , and CE to AT , then the triangles ADC and $A'T'B'$ have $AC = A'B'$, $\angle DAC = \angle T'A'B'$, and $\angle DCA = \angle T'B'A'$ (each being equal to $\angle TBA$). So that the triangles are congruent. And so on. The lengths AD , $A'T'$, CE , DT are all equal, and therefore $A'T'$ is half of AT .

12. Let us now denote by AB the distance between A and B in some chosen unit, say in centimetres; and so with every two points, so that LN means the distance apart of L and N in centimetres. Let us denote the actual position of the tree by t and the actual ends of the fence by a and b .

Suppose that our former chain surveying gave the distance a to b 37·3 metres, distance a to t 30·5 metres, distance b to t 14·7 metres. [The teacher should not use these values, but values resulting from an actual survey by the class.] With the above meaning for ab , &c., we have $ab = 3730$, $at = 3050$, $bt = 1470$. What are the values of the fractions $\frac{AB}{ab}$, $\frac{A'B'}{ab}$, $\frac{XY}{ab}$, $\frac{LM}{ab}$?—These are other

ways of writing the fractions we had before, $1/500$, $1/1000$, $3/1000$, $1\cdot8/1000$, or $0\cdot002$, $0\cdot001$, $0\cdot003$, $0\cdot0018$.

Suppose that measurement gives $\angle tab = 22^\circ$ and $\angle tba = 52^\circ$. By means of the measured length of the fence, and these angles, make the four plans of the fence and tree. Measure up these plans and calculate as decimals

$\frac{AT}{at}$, $\frac{A'T'}{a't'}$, $\frac{XZ}{xz}$, $\frac{LN}{ln}$, $\frac{BT}{bt}$, $\frac{B'T'}{b't'}$, $\frac{YZ}{yz}$, $\frac{MN}{mn}$. Further, from these measurements and the values of AB , $A'B'$, &c., which were used to make the plans, calculate as decimals $\frac{AB}{AT}$, $\frac{A'B'}{A'T'}$, $\frac{XY}{XZ}$, $\frac{LM}{LN}$, and compare them. Do the same with $\frac{BT}{BA}$, $\frac{B'T'}{B'A'}$, $\frac{YZ}{YX}$, $\frac{MN}{ML}$.

The fractions dealt with in this article have a special name ratio; $\frac{MN}{ML}$ with its full title is 'the ratio of MN to ML '.

13. Draw any two lines intersecting in O and across them draw a series of parallels, and letter the figure as in Fig. 5.

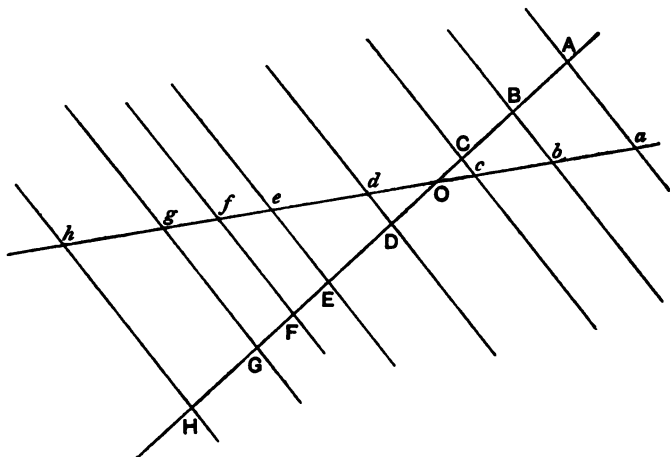


FIG. 5.

Measure up the figure and calculate as decimals the ratios or fractions $\frac{OA}{Oa}$, $\frac{OB}{Ob}$, $\frac{OC}{Oc}$, $\frac{OD}{Od}$, &c., and compare them.

Compare also $\frac{OA}{OB}$ with $\frac{Oa}{Ob}$, $\frac{OA}{OC}$ with $\frac{Oa}{Oc}$, &c. State formally your experimental result. Can you prove the result by general reasoning?—Suppose that we find $OA = 3.6$

and $OB = 2.1$, the centimetre being the unit. We mark OP along OA equal to 0.1 cm. and draw Pp parallel to Aa and Bb to cut Oa in p . Then OA is 36 times OP , and we can, as before, compare the triangles OAA and OPp and show Oa to be 36 times Op . Also OB is 21 times OP , and by comparing Δs OBb and OPp we can show Ob to be 21 times Op . We had $\frac{OA}{OB} = \frac{3.6}{2.1} = 1.7$; and now we have $\frac{Oa}{Ob} = \frac{36}{21} = 1.7$, since Oa is 36 times Op and Ob is 21 times Op .

14. Suppose OA and OB not to be exactly of the lengths given, but to be 3.58 and 2.13 when more accurately measured. How would you then prove that the fractions or ratios $\frac{OA}{OB}$ and $\frac{Oa}{Ob}$ are equal?—We should then have to make (or to imagine made) OP 0.01 cm. long, so that OA would be 358 times OP and OB would be 213 times OP . The proof runs as before, and from our more accurate measurements we get a more accurate value for the fractions OA/OB and Oa/Ob , namely $358/213$ or 1.68 . Give the proof in detail for this case.

If by greater refinement in measuring we could measure OA and OB to 3 decimal places, or to 4 decimal places, how would you proceed with the comparison of these ratios or fractions?—We choose OP of an appropriate length, and the proof runs as before.

Is it possible to measure a length with absolute accuracy, or should you expect every increase of refinement to add to the number of digits we must use for the expression of the length?—The latter seems likelier.

Then to what extent is your proof of the equality of OA/OB and Oa/Ob valid?—It is valid only if it is agreed to pay no attention to errors of a certain degree of smallness.

Horse-dealing.

15. That it is unsafe to neglect small things is well illustrated by an old story. Two men were negotiating the sale of a horse, and the seller offered to give the horse provided the horseshoe nails were bought at the rate of 1 farthing for the first, 2 farthings for the second, and so on, the price doubling with each nail. The buyer jumped

at the offer. What was the price of the last nail, there being 8 nails in each shoe?—The price of the last nail was more than £2,000,000. So fast does the small sum of one farthing multiply.

By unaided arithmetic this calculation is very heavy. The price of the last nail in farthings is $2 \times 2 \times 2 \times \dots \times 2$, there being 31 2's in the product. A short way of writing such a product is a considerable help. The product in question may be written 2^{31} . With this notation the prices of the 1st, 2nd, 3rd, 4th . . . nails are in farthings 1, 2, 2^2 , 2^3 , . . . The number written above and to the right to indicate the number of 2's is called the index, and the product so expressed is called a power; thus 2^{31} is called 'the thirty-first power of 2', and its index is 31.

16. Can you suggest any shorter way of calculating 2^{31} than to begin with 2 and double the necessary number of times?—We could divide the 31 2's into two lots of 16 and 15. If we then multiply together the 15 2's, we can get the product of the 16 2's by doubling once more; and these two products multiplied together will give the product of 31 2's. With the index notation this relation may be written $2^{15} \times 2^{16} = 2^{31}$.

Can you shorten the calculation any further?—We could divide the 15 2's into batches, say into three batches of 5 2's; we could calculate the product of 5 2's, and then three such products multiplied together give the product of 15 2's. With our notation $2^5 \times 2^5 \times 2^5 = 2^{15}$. You have now a fairly quick way of calculating 2^{31} . Use it to calculate this quantity roughly, say by stopping in each product of the process at two significant figures.—

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32;$$

$$2^{15} = 32 \times 32 \times 32 = 33,000 \text{ approximately};$$

$$2^{16} = 33,000 \times 2 = 66,000;$$

$$2^{31} = 33,000 \times 66,000 = 2,200,000,000.$$

And this being the number of farthings, the price of the last horseshoe nail is £2,300,000. The second figure is unreliable, so the price may be given roughly as £2,000,000.

17. Perhaps the quickest way is to calculate first 2^{32} . We can divide the 32 2's into two batches of 16 each,

then each of these into batches of 8 each, and again each of these into batches of 4 each. Now

$$2^4 \text{ or } 2 \times 2 \times 2 \times 2 = 16;$$

so that a batch of 8 2's or 2^8

$$= 16 \times 16 = 260;$$

a batch of 16 2's or 2^{16}

$$= 260 \times 260 = 68,000;$$

and the batch of 32 2's or 2^{32}

$$= 68,000 \times 68,000 = 4,600,000,000,$$

and as this contains a 2 too many, we divide by 2 and get

$$2^{31} = 2,300,000,000,$$

the difference in the second place between this and the former result being due to the figures discarded.

18. From the way the price of horseshoe nails mounts up the possibility is suggested that, by neglecting a small quantity by which OA/OB (Fig. 5) differs from the value we use for it, we may go seriously wrong in the value we

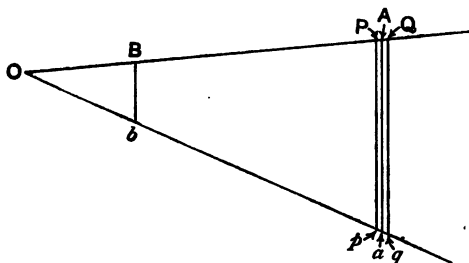


FIG. 6.

give Oa/Ob . Let us try to find out how great an error there can be in the deduced value of Oa/Ob .

To take a definite case, let us suppose that OA has the length of the circumference of the circle drawn on OB as diameter (Fig. 6). We cannot measure the circumference of a circle very accurately, but the circumference for a given diameter can be calculated to any degree of accuracy. It is known that OA is greater than 3.141,592,653 times OB , and less than 3.141,592,654 times OB . It is known further that

no number of decimal places will express the ratio with complete accuracy. Let us for shortness denote 3,141,592,653 by m , and 1,000,000,000 by n . Then OA is greater than $\frac{m}{n}$ times OB , and less than $\frac{m+1}{n}$ times OB . On OBA cut off OP equal to $\frac{m}{n}$ times OB , and OQ equal to $\frac{m+1}{n}$ times OB . Thus A lies between P and Q . And Aa , Bb , Pp , and Qq being all drawn parallel (Fig. 6), the point a lies between p and q .

19. If now we suppose OB divided into n equal parts, the n th part of OB can be laid off an exact number of times along OP , namely m times; and also an exact number of times along OQ , namely $m+1$ times. Our former reasoning therefore applies and shows that Op is $\frac{m}{n}$ times Ob , and Oq is $\frac{m+1}{n}$ times Ob . Hence Oa , which is intermediate in length between OP and OQ , lies between $\frac{m}{n}$ and $\frac{m+1}{n}$ times Ob , that is, between 3·141,592,653 and 3·141,592,654 times Ob . In this case then the small error has not multiplied.

Further, we could have begun with a value giving OA/OB to a hundred decimal places, and the form of reasoning would need no change. So that whatever degree of accuracy we consider necessary for OA/OB , we know that to that degree of accuracy Oa/Ob has the same value.

Remembering that OA , OB , Oa , Ob are mere numbers, namely, the numbers of centimetres in the lengths of the lines, we can state the result approximately in any of the forms

$$\begin{aligned}\frac{OA}{OB} &= \frac{Oa}{Ob}, \\ OA \times Ob &= OB \times Oa, \\ \frac{OA}{Oa} &= \frac{OB}{Ob}, \\ \frac{Oa}{OA} &= \frac{Ob}{OB}, \text{ \&c.}\end{aligned}$$

20. * In the preceding article we found that approximately $OA/OB = Oa/Ob$. Now we cannot by means of decimals express the lengths OA , &c., accurately; for these as well as for the ratios we must use approximations. Do all these small errors mount up so that the other forms, $OA \times Ob = Oa \times OB$, &c., which we deduce from $OA/OB = Oa/Ob$, contain serious errors, or are the errors in these forms also small? Suppose the lengths of all these lines taken in centimetres, to a millionth of a centimetre, so that $OA = p$ approximately if not accurately, and that if p is not the accurate value it is within a millionth of it. Thus OA is greater than $p - 0.000,001$ and less than $p + 0.000,001$; or, if for shortness we write k for $0.000,001$, and use the symbol $>$ to mean 'is greater than', and $<$ to mean 'is less than',

$$p - k < OA < p + k.$$

Suppose q, r, s the lengths of the other lines to the same approximation, so that

$$q - k < OB < q + k,$$

$$r - k < Oa < r + k,$$

$$s - k < Ob < s + k.$$

Let us try to find by how much $\frac{p}{q}$ can differ from $\frac{OA}{OB}$, $\frac{r}{s}$ from $\frac{Oa}{Ob}$, $\frac{p}{q}$ from $\frac{r}{s}$, $p \times s$ from $q \times r$, &c.

21. The greatest possible value of the fraction $\frac{OA}{OB}$ will be found by giving OA its greatest value $p + k$, and OB its

* This discussion (articles 20-24, 26, 27, 29) is too difficult for the present stage. Let the pupils assume that the neglect of a small quantity will only cause a small error in the result. Let them proceed on this assumption to discuss the statements at the end of art. 24 and to work through arts. 25 and 28, and then let them go on to art. 30. At the end of the course contained in this book the pupils could with profit return to the discussion given here.

least value $q-k$; it is $\frac{p+k}{q-k}$. To find how much this differs from $\frac{p}{q}$, we bring the two fractions to the same denominator $q \times (q-k)$ in the usual arithmetical way. The two numerators are then $q \times (p+k)$ and $(q-k) \times p$, or $qp + qk$ and $qp - kp$, and their difference is $qk + pk$ or $k \times (q+p)$. The difference between the fractions is therefore $\frac{k \times (p+q)}{q \times (q-k)}$, that is, it is one millionth of $\frac{p+q}{q(q-k)}$.

In the same way show that the least possible value of the fraction $\frac{OA}{OB}$ differs from $\frac{p}{q}$ by one millionth of $\frac{p+q}{q(q+k)}$.

Since $q+k$ is greater than $q-k$, the fraction $\frac{p+q}{q(q+k)}$ is less than the fraction $\frac{p+q}{q(q-k)}$, so that $\frac{OA}{OB}$ and $\frac{p}{q}$ do not differ by more than a millionth of $\frac{p+q}{q(q-k)}$, or, in the shorter form already used, by more than $\frac{k(p+q)}{q(q-k)}$.

In the same way show that $\frac{Oa}{Ob}$ and $\frac{r}{s}$ do not differ by more than $\frac{k(r+s)}{s(s-k)}$.

22. Knowing that $\frac{OA}{OB}$ and $\frac{Oa}{Ob}$ are equal, what can you tell about the possible difference between the approximate values $\frac{p}{q}$ and $\frac{r}{s}$?—Since $\frac{p}{q}$ does not differ from these equal fractions by more than $\frac{k(p+q)}{q(q-k)}$, nor $\frac{r}{s}$ by more than $\frac{k(r+s)}{s(s-k)}$, $\frac{p}{q}$ and $\frac{r}{s}$ cannot differ by more than

$$\frac{k(p+q)}{q(q-k)} + \frac{k(r+s)}{s(s-k)},$$

that is, than one millionth of

$$\frac{p+q}{q(q-k)} + \frac{r+s}{s(s-k)}.$$

This last quantity is calculated from ordinary lengths $p+q$, q , $q-k$, $r+s$, s , $s-k$, and so is of an ordinary size. Let us denote it by N for shortness; so that $\frac{p}{q}$ and $\frac{r}{s}$ do not differ by more than kN , that is, a millionth of N .

23. By how much can the quantities ps and qr differ?— ps is qs times $\frac{p}{q}$, and qr is qs times $\frac{r}{s}$; so that the difference between ps and qr is qs times the difference between $\frac{p}{q}$ and $\frac{r}{s}$, which is not more than kN . So that ps and qr do not differ by more than $qs \times kN$, that is, a millionth of qsN . That is, when we use the approximate values p , &c., for OA , &c., the relation $OA \times Ob = OB \times Oa$ is true to a pretty close approximation.

What is the greatest value $\frac{p}{r} - \frac{q}{s}$ can have? $\frac{p}{r} - \frac{q}{s}$ is $\frac{q}{r} \times (\frac{p}{q} - \frac{r}{s})$, so that the greatest value of the difference is $\frac{q}{r} \times kN$, or one millionth of $\frac{qN}{r}$. That is, with the approximate values of OA , &c., $\frac{OA}{Oa}$ is equal to $\frac{OB}{Ob}$ to a pretty close approximation.

In the same way show that $\frac{Ob}{OA} = \frac{Oa}{OB}$ pretty closely, with the approximate values for OA , &c.

24. Further, we could use closer arithmetical values for OA , &c., and find correspondingly smaller possible differences between $OA \times Ob$ and $OB \times Oa$, between OA/Oa and OB/Ob , &c. We can make these differences as small as we please by taking close enough approximations for OA , &c.

Again, the reasoning has not depended on the special relation between OA and OB (that OA is π times OB), so that the general result may be stated in any of the following forms:—

A parallel to the base of a triangle divides the

two sides in the same ratio. For example, in Fig. 7 $AD/AB = AE/AC$.

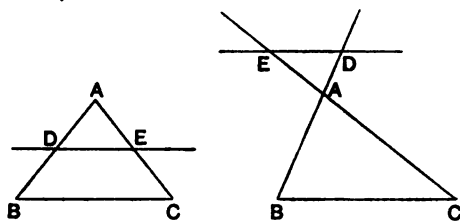


FIG. 7.

If two triangles have the three angles of the one equal to the three angles of the other, corresponding sides of the two triangles have the same

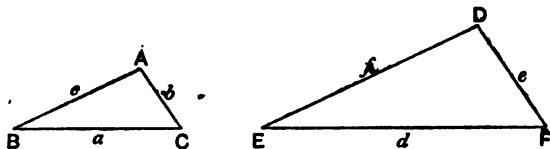


FIG. 8.

ratio. For example, in Fig. 8, in which $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$, we know that

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f},$$

or, otherwise stated,

$$\frac{a}{b} = \frac{d}{e}, \quad \frac{b}{c} = \frac{e}{f}, \quad \frac{c}{a} = \frac{f}{d}.$$

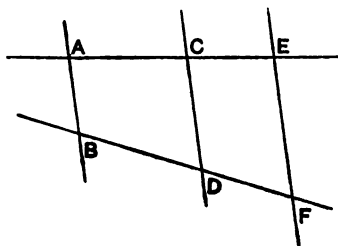


FIG. 9.

Three parallel lines falling across two other lines divide them in the same ratio. For example, in Fig. 9

$$\frac{AC}{CE} = \frac{BD}{DF}.$$

Ex. 9. The third form of statement just given has not been completely proved. Supply the missing point; for instance, by drawing through A a parallel to BF .

25. Ex. 10. As a test of the statements of the preceding article calculate all the ratios mentioned as decimals from measurements of the figures.

Ex. 11. Fig. 10 shows a sextant, an instrument for measuring angles. The frame ABC carries an arm AD pivoted at A . This arm carries at the end A a mirror E set at right angles to the frame. F is another mirror perpendicular to the frame and parallel to AB . To measure the angle between two lines running from the observer's position to two objects P and Q at the same level as the observer (in other words, the angle subtended by P and Q at the observer's position), the observer holds the sextant up horizontally so that, with his eye at the point G marked on the bar AB , he sees P over the mirror F . He then turns the arm AD until he sees Q in the mirror F , the ray of light from Q coming to his eye by reflection at E and then at F . It is known that when a ray of light falls on a mirror and is reflected, the ray falling on the mirror and the reflected ray are in the same plane with the perpendicular to the mirror and make equal angles with it. The point G is taken so that when the arm AD lies along AB and the observer looks at P over the mirror F , he sees P also in the mirror by reflection at E and F .

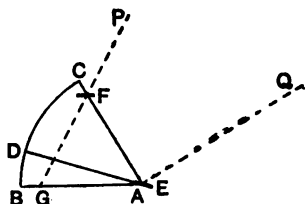


FIG. 10.

Show how the point G may be fixed on the bar AB . And show that the angle subtended by P and Q is twice the angle BAD . Each object is so far away that all lines from the frame to it may be considered parallel.

Ex. 12. Use a sextant to measure the angles for the purpose of drawing the plan of the playground.

26. Let us now return to the plans of the playground made by a measuring-tape or a surveyor's chain only. We found corresponding angles of all the plans to be equal.

But our proof depended on the commensurability of the lengths involved, that is on the fact that a unit could be chosen in terms of which the lengths could be expressed accurately.

Suppose we have to do with two triangles ABC and DEF (Fig. 11), in which the lengths of the sides are incommensurable, that is, such that no unit can be chosen in terms

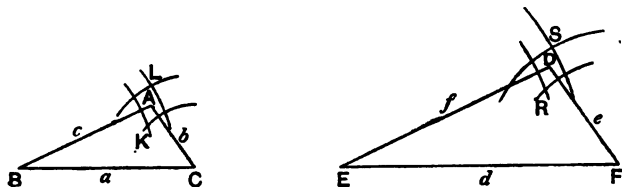


FIG. 11.

of which the lengths can be expressed. Suppose that, in whatever units and to whatever degree of approximation the lengths a, b, c, d, e, f of the sides are taken, the fractions or ratios $\frac{c}{a}$ and $\frac{f}{d}$ are equal, and $\frac{b}{a}$ and $\frac{e}{d}$ are equal. Can we prove corresponding angles of these triangles to be equal?

Let us imagine BC divided into a great number n of equal parts, for instance a million, and imagine EF also divided into n equal parts. Let us imagine that BA contains more than m and less than $m+1$ of these small parts of BC ; from the equality of the fractions $\frac{c}{a}$ and $\frac{f}{d}$ we know that ED contains more than m and less than $m+1$ of the small parts of EF . Imagine further that AC contains more than p and less than $p+1$ of the small parts of BC , and consequently DF contains more than p and less than $p+1$ of the small parts of EF .

Let us imagine made on BC and EF triangles KBC and REF for which $\frac{c}{a}$ and $\frac{f}{d}$ are equal to $\frac{m}{n}$, and $\frac{b}{a}$ and $\frac{e}{d}$ equal to $\frac{p}{n}$. Their sides being commensurable we know how to prove their corresponding angles equal. And these triangles do not differ much from the triangles ABC and DEF .

27. If we had taken the values $\frac{m+1}{n}$ and $\frac{p+1}{n}$ instead of $\frac{m}{n}$ and $\frac{p}{n}$, or if we had taken $\frac{m+1}{n}$ and $\frac{p}{n}$, or $\frac{m}{n}$ and $\frac{p+1}{n}$, we should have found other triangles differing little from ABC and DEF , and having corresponding angles equal.

And by taking a great enough value for n we can make these triangles with corresponding angles equal differ by as little as we please from the triangles ABC and DEF . We can thus make two triangles KBC and REF which have $\angle KBC$ and $\angle REF$ equal, $\angle KBC$ and $\angle ABC$ differing by as little as we please, and $\angle REF$ and $\angle DEF$ differing by as little as we please. This shows that $\angle ABC$ and $\angle DEF$ differ by less than a quantity that we can make as small as we please. So that to whatever approximation we go, $\angle ABC = \angle DEF$, and similarly, $\angle BCA = \angle DFE$, and $\angle BAC = \angle EDF$.

The equality of the ratios of the sides of the triangles was used in the form $\frac{c}{a} = \frac{f}{d}$, $\frac{b}{a} = \frac{e}{d}$. What if the equality had been given in the form $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$?—We know that from this form it follows that, to any approximation we please, $\frac{c}{a} = \frac{f}{d}$ and $\frac{b}{a} = \frac{e}{d}$.

28. The triangles ABC and DEF are called similar when the corresponding sides have the same ratio and the corresponding angles are equal, that is, when $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$. Further, any two figures (whether triangles or not) are called similar when corresponding lines on them have the same ratio and corresponding angles are equal.

In what circumstances have you already proved two triangles similar?—It has been proved when two pairs of angles are equal, e.g. $\angle B = \angle E$ and $\angle C = \angle F$; and when there are two pairs of equal ratios among the corresponding sides, e.g. when $\frac{c}{a} = \frac{f}{d}$ and $\frac{b}{a} = \frac{e}{d}$.

Can you suggest any other circumstances that are enough to prove the triangles similar; or, in other words, can you give any other set of data that are enough to fix the shape of a triangle?—Possibly the triangle would be fixed in shape by one angle and the ratio of one pair of sides. Possibly two triangles, with a pair of corresponding angles equal, and one equal ratio among corresponding sides, might be similar.

Discuss all the possible forms of these conditions; first draw as many as you can of the triangles determined in shape by the angle B and one ratio, giving B in all cases the value 35° , and taking as the ratio each of the following in succession: $\frac{c}{a} = 0.7, 1.4, 4; \frac{a}{c} = 1.8,$

$0.5, 3; \frac{a}{b} = 1.6, 2.0, 0.4; \frac{b}{a} = 0.6, 0.4, 1.5; \frac{b}{c} = 0.8,$

$0.5, 2; \frac{c}{b} = 1.2, 2.1, 0.9.$ —Drawing gives the following

experimental results. There is one shape of triangle possible

when B and $\frac{c}{a}$ (or $\frac{a}{c}$) are given. There is one shape

when B and the ratio of b to another side are given if this given ratio is greater than 1. There are two shapes or no shape when B and the ratio of b to another side are given if the ratio is less than 1.

29. Show that if two triangles ABC and DEF (Fig. 12)

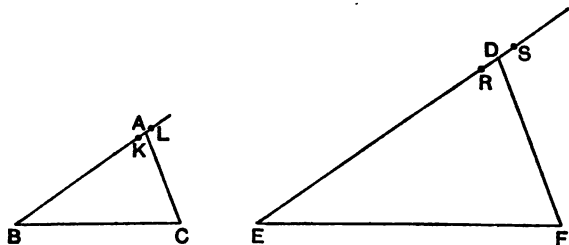


FIG. 12.

have the angles B and E equal and the ratios $\frac{c}{a}$ and $\frac{f}{d}$ equal, they have also the other corresponding angles equal,

and so are similar. Assume the ratios $\frac{c}{a}$ and $\frac{f}{d}$ to lie between $\frac{m}{n}$ and $\frac{m+1}{n}$; along BA mark off $BK = \frac{m}{n}$ times BC , and $BL = \frac{m+1}{n}$ times BC ; and so for $\triangle DEF$. Then proceed as before.

Show that if two triangles ABC and DEF have $B = E$ and $\frac{b}{c} = \frac{e}{f}$, each ratio being greater than 1, the two triangles are similar. Where does this proof depend on the condition that the ratios $\frac{b}{c}$ and $\frac{e}{f}$ are greater than 1?—In the course of the proof we cut off small equal lengths BW from BA and EY from ED (Fig. 13), and draw WX

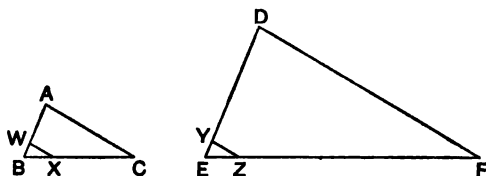


FIG. 13.

parallel to AC , and YZ parallel to DF . The condition that $\frac{b}{c}$ and $\frac{e}{f}$ are greater than 1, shows that $WX > WB$ and $YZ > YE$, and in consequence that the triangles BWX and EYZ are congruent.

30. Now state generally what conditions fix the shape of a triangle:—

- A triangle is fixed as to shape
- (1) by a knowledge of two angles, or
 - (2) by a knowledge of two ratios of its sides, or
 - (3) by a knowledge of one angle and of the ratio of the two sides that form this angle, or
 - (4) by a knowledge of one angle and of the ratio the opposite side bears to another side provided this ratio is greater than unity.

Or, in other words, two triangles are similar

(1) if they have two pairs of corresponding angles equal; or

(2) if they have the ratios of two sides to the third in one triangle equal to the corresponding ratios in the other; or

(3) if they have a pair of corresponding angles equal, and the ratios of the sides forming these angles also equal; or

(4) if they have a pair of corresponding angles equal and the ratio of the side opposite this angle of one triangle to a second side equal to the corresponding ratio in the other triangle, provided these ratios are greater than 1.

31. When a triangle is to be made from $B = 35^\circ$ and a value of $\frac{c}{b}$, for what values of $\frac{c}{b}$ are there two shapes, for what values one, and for what values none?—We lay down the angle B and put down the point A anywhere on one arm (Fig. 14). We then calculate the value of b ,

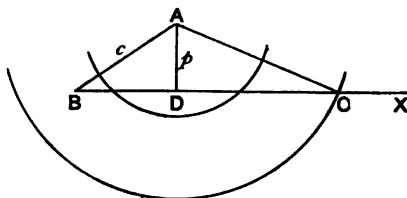


FIG. 14.

and with this as radius draw a circle which cuts the other arm BX in C . If $\frac{c}{b}$ is less than 1, this circle cuts BX in only one point, and there is only one shape. If $\frac{c}{b}$ is greater than 1, the circle (if great enough to meet BX at all) cuts it in two points and gives two shapes. Let fall the perpendicular AD or p from A on BX . If the given ratio $\frac{b}{c}$ is equal to $\frac{p}{c}$, the circle cuts BX in only one

point ; if it is less, in none ; if it is greater than $\frac{p}{c}$ but less than 1, in two points.

Does this ratio $\frac{p}{c}$ depend on the particular point at which A was put down on the arm of the angle B ?—No, for the triangle ABD is fixed in shape by the angles B and D , namely $B = 85^\circ$ and $D = 90^\circ$. Two triangles fixed by these angles are similar and so have the same value for the ratio $\frac{p}{c}$. Draw a number of such triangles ABD , measure p and c in each and so find the value $\frac{p}{c}$ as accurately as you can.—The value is about 0·57.

Tabulate the results as to the possible shapes of a triangle with an angle B of 35° , and various values of $\frac{b}{c}$.—

If $\frac{b}{c} > 1$, there is one shape,

if $1 > \frac{b}{c} > 0\cdot57$, there are two shapes,

if $\frac{b}{c} < 0\cdot57$, there is no shape.

Ex. 13. Find and tabulate the corresponding results for $\angle B = 25^\circ, 30^\circ, 40^\circ, 45^\circ$.

32. It is convenient to have a name for this ratio $\frac{p}{c}$ (Fig. 14) which depends only on the angle B . It is called the 'sine of the angle B ' and written shortly ' $\sin B$ '. Thus we have found that $\sin 35^\circ = 0\cdot57$ as nearly as our drawing will give it. A book of mathematical tables gives the result more accurately as the result of calculation ; the value of $\sin 35^\circ$ taken to 7 decimal places is 0·5735764.

Ex. 14. From the results of Ex. 13 write down the sines of $25^\circ, 30^\circ, 40^\circ, 45^\circ$, and beside them show more accurate values from a book of tables.

Ex. 15. Draw four right-angled triangles in which one side is 0·2, 0·4, 0·6, 0·8 times the hypotenuse. From these find the

angles whose sines are 0·2, 0·4, 0·6, 0·8, and verify your results by reference to the book of tables.

33. Areas of similar figures. Consider again Figs. 1–4, which we now know to be similar. What is the ratio $\frac{AB}{A'B'}$, and what is the ratio of the area ABT to the area $A'B'T'$, that is, how many times does the area ABT contain the area $A'B'T'$? Pair the four figures in all possible ways, and for each pair give the ratio of the bases and the ratio of the areas.

If two similar triangles have a pair of corresponding sides in the ratio k , find the ratio of their areas.—Suppose the similar triangles of Fig. 15 to have $\frac{BC}{EF} = k$. Draw the

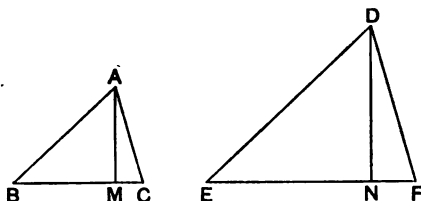


FIG. 15.

perpendiculars AM and DN . The areas of the triangles are $\frac{1}{2} \times AM \times BC$ and $\frac{1}{2} \times DN \times EF$. $BC = k \times EF$, and $AM = k \times DN$. So that the first area is $\frac{1}{2} \times k \times DN \times k \times EF$; or $k \times k$ times $\frac{1}{2} \times DN \times EF$. The ratio of the areas is $k \times k$, or with the index notation it is k^2 . Prove that

$AM = k \times DN$, that is, that $\frac{AM}{DN} = \frac{BC}{EF}$.

34. If two plans of the playground are made by chain surveying, are they similar?—Yes, because each plan is made triangle by triangle, and taking pairs of corresponding triangles in succession we prove their angles equal. Thus the whole figure conforms to our definition of similarity.

And if the two plans were made by the measurement of angles only in addition to one base line? The proof that they are similar is much the same.

If the two base lines of the plans, made by either

method, have the ratio k , what is the ratio of the two areas that represent the playground?—Each triangle of one plan is as to area $k \times k$ times the corresponding triangle. So that the whole area of the first is $k \times k$ times the area of the second.

Is the result affected if the playground has a curved boundary?—By taking triangles enough we can reduce as much as we like the area along the curved boundary with which we cannot deal. So that we can prove the result true to any approximation we like.

35. *Volumes of similar solids.* If a schoolroom is p cm. long, q cm. broad and r cm. high, what is its volume?—It is $p \times q \times r$ cubic centimetres.

And if a model of the schoolroom is made, each dimension being k times the original dimension, so that its length, breadth, and height are in centimetres $k \times p$, $k \times q$, and $k \times r$, what is the volume of the model?—It is $k \times p \times k \times q \times k \times r$ c. cm., or (omitting the signs of multiplication) $kpkqkr$ or $kkk pqr$, or (with the index notation) $k^3 pqr$.

What ratio does the model bear to the room as to volume?—The ratio is k^3 .

The original room and the model are said to be similar, and in general two solid figures in which corresponding angles are equal and corresponding lines in the same ratio are said to be similar. If this ratio is k , what is the ratio of the volumes? For instance, suppose a model of a mountain to be made, every distance on the model being k times the original distance, and the two being assumed similar.—Suppose the mountain cut up into rectangular blocks and the model into corresponding blocks. Then each block of the model is kkk times the corresponding block of the mountain. So that, apart from small pieces left over that do not make rectangular blocks, the ratio is kkk . And by cutting out smaller blocks we can make the piece left over as small as we like, so that the ratio of the whole model to the whole mountain is kkk or k^3 , to as close an approximation as we please.

36. How many conditions fix the shape of a triangle?—Two, as we saw in Art. 30.

How many conditions fix the triangle as to size and shape?—One length is needed in addition, for instance the length of a side. Or considering size and shape together, we saw earlier (chap. V,

art. 27) that three conditions fix the triangle. How many conditions fix the triangle in shape, size, and position on a sheet of paper?—This brings us back to the problem of fixing a chair on the floor (chap. II). To fix the position takes three conditions, so that shape, size, and position take six conditions. Or we can consider shape, size, and position all at once. For instance, these are all fixed by giving the distances of each corner from two adjacent edges of the sheet of paper, in all six conditions. Suggest other ways of fixing the triangle in shape, size, and position, and see if the number of conditions is six in all cases.

EXERCISES.

16. Draw a line AB 3·8 inches long, and AC of indefinite length making with it an angle of 60° . On AC mark off AK , KL , LM each of length 1·5 inches. Join MB and draw KX and $LY \parallel$ to MB , cutting AB in X and Y .

Show by measurement that by this construction AB is divided into three equal parts.

Give also a general proof, showing that the construction would hold if AC were drawn at some other angle to AB , and some other length taken instead of 1·5 inches. For the general proof you may find it convenient to draw lines through X and Y parallel to AC .

17. Make a triangle, ABC , having $\angle A = 90^\circ$ and $\angle B = 18^\circ$ and AB 14 cm. long, and let fall AM perpendicular to BC . Measure CA , CB , AM , CM , and calculate the ratios $\frac{CA}{CB}$, $\frac{AM}{AB}$, $\frac{CM}{CA}$ to three significant figures.

It is known that, if the figure could be perfectly drawn and measurements taken with absolute accuracy, each of these ratios would have the value $\frac{\sqrt{5}-1}{4}$. By means of a table of squares

evaluate this to three figures, and compare it with the average of the values already found.

Show briefly, that if a number of triangles ABC are made, in all of which A and B have the same given values, then the ratio CA/CB is the same in all these triangles. State, with reasons, whether the ratio of the angle B to the angle A is the same as the ratio CA/CB .

18. An enlarged photograph of Fig. 16 is to be made, bc being represented by a line BC 15 centimetres long. Copy Fig. 16, and then draw the enlargement, denoting the points corresponding to a b c &c., by A B C &c.

Find, from measurement, the area of the rectangle df in sq. cm., also of DF , and give de/DE , df/DF as decimals.

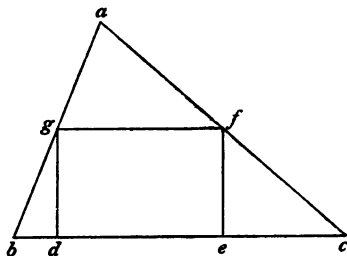


FIG. 16.

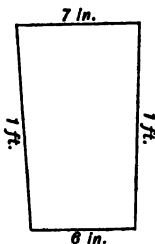


FIG. 17.

19. A semicircular arch is built of stones cut to the shape shown in Fig. 17. What is the internal diameter of the arch, and how many stones are required to complete it?

20. Draw any circle and estimate its area. Then test your estimate by measurement and calculation, taking the area as $3.14 \times (\text{radius})^2$, and also by pricking through on to squared paper and counting squares.

21. I live a mile from my railway station and take 16 minutes to walk the distance. I am accustomed to meet at a certain point some children going in the opposite direction to school. The children take 25 minutes to the mile. One day when I am at my usual time I meet them 200 yards further on. How much later are the children than on ordinary days? Draw a graph to show the position of the children and myself at any time on ordinary days and on the special day mentioned.

22. A hay-shed (Fig. 18) consists of a rectangular space 30 feet long, 15 feet high and 18 feet wide, with a semicircular roof above it. Calculate how many tons of hay can be stored in it if a ton of hay occupies 300 cubic feet.

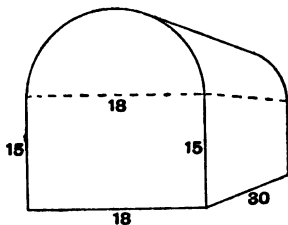


FIG. 18.

23. Draw two congruent triangles ABC and XYZ , such that the points A, B, C and the points X, Y, Z are in the same direction round. Find a point S such that $SB = SY$ and $SC = SZ$. Prove that the angle $SCB = \text{angle } SZY$ and $SA = SX$.

Suppose a piece of tracing-paper is laid on the figure and fastened by one drawing pin so that it can revolve round this drawing pin

while remaining in contact with the paper; suppose also that when the tracing-paper is in its first position a tracing is made of the triangle ABC . Where must the drawing pin be placed if it is found that as the tracing-paper revolves, the tracing of ABC presently coincides with XYZ ?

24. Compute to two significant figures, the diameter and cross sectional area of a specimen of copper wire, one mile of which weighs 4 lb. One cubic inch of copper weighs 0.32 lb.

25. Two quantities x and y are so related that the value of y for a given value of x may be found as follows :—Along OX (Fig. 19)

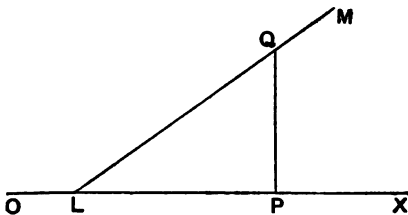


FIG. 19.

measure OP (in centimetres) equal to x , and draw PQ perpendicular to OX to meet LM in Q ; then $PQ = y$. Copy the figure and find the values of y when $x = 2, 3, 4$ and 5 .

Measure OL , and in each case calculate $\frac{y}{x - OL}$ to one decimal place. State the relation between x and y algebraically.

CHAPTER X

ON SIGNALLING

1. In the Morse Code, which is a good deal used in postal telegraphy and other methods of signalling, there are two signs, the dot and the dash. Each letter of the alphabet is represented by some arrangement of dots and dashes. How many arrangements can be made that contain three signs or less?—We set out the arrangements, 1 sign at a time, 2 at a time, and 3 at a time, and count them up :—

.	—		
. .	. —	— .	— —
. . .	. . —	. — —	— — —
	. — .	— . —	
	— . .	— — .	

With one sign at a time there are 2 arrangements, with 2 signs 4, with 3 signs 8 ; in all 14.

2. How many arrangements are there with 4 signs? Haphazard counting soon becomes tedious and even inaccurate, and for expeditious and accurate counting some system is necessary. How would you convert a 3-arrangement into a 4-arrangement, or a 2-arrangement into a 3-arrangement? Does it matter (for the purpose of counting the 4-arrangements) whether you add the sign at the beginning, at the end, or somewhere in the middle?—By taking a 3-arrangement and inserting a dot we get different 4-arrangements according to the position of the inserted dot, but if we treat all 3-arrangements in this way the same 4-arrangement turns up in more than one way. For instance, . — . with a dot inserted at the beginning is the same as . . — with a dot inserted at the end. But if we always add the dot at the end each 3-arrangement gives one

4-arrangement, and there is no repetition. By adding a dash at the end instead of a dot we get another 4-arrangement from each 8-arrangement. So that there are twice as many arrangements of 4 signs as of 8 signs.

3. In this way tabulate the number of arrangements of 1, 2, 8, 4 signs.—

The number of 1-arrangements is 2.

“ “ 2 “ “ 2×2 or 4.

“ “ 3 “ “ $2 \times 2 \times 2$ or 8.

“ “ 4 “ “ $2 \times 2 \times 2 \times 2$ or 16.

How many 10-arrangements are there?—The number is ten 2's multiplied together (which is also written 2^{10}), and is 1024.

How many n -arrangements?—The number is n 2's multiplied together, which we write shortly as 2^n .

How many signs are there in the longest arrangements of a code that represent letters of the alphabet, the code being made with as short arrangements as possible?—To stop at 8-arrangements would give only 14 letters. To go to 4-arrangements gives us the 26 letters of the alphabet and 4 arrangements to spare.

4. Consider again the building up of an arrangement of 4 signs. What choice have you for the first sign? And the first sign being chosen, what choice have you for the second? In how many ways then can the first two signs be chosen? Continuing in this way discuss the possible number of arrangements of 4 signs, of 10 signs, and of n signs.

5. How many arrangements can be made without going beyond 10 signs for any one arrangement?—We must add the numbers for 10 signs, 9 signs, &c. It is $1024 + 512 + 256 + 128 + 64 + 32 + 16 + 8 + 4 + 2$ or 2046.

For arrangements of more signs than 10, this method would be very heavy. Let us denote by N the number of arrangements that can be made without going beyond 10 signs. Now write down twice the number of 10-arrangements, twice the number of 9-arrangements, and so on, whose sum is $2N$.—These numbers are $2048 + 1024 + 512 + 256 + 128 + 64 + 32 + 16 + 8 + 4$.

Can you subtract the expression found for N from the one now found for $2N$, without adding up either of them? What is the result?—It is $2N - N = 2048 - 2$,

since every number but 2048 and 2 occurs in both expressions. This gives $N = 2046$.

6. 2048 being the product of 11 2's we can write the value of N as $2^{11}-2$. Can you prove the correctness of this expression without multiplying out 2^{11} , 2^{10} , &c.?—The number of 10-arrangements is the product of 10 2's or 2^{10} , and so with each other set of arrangements, so that

$$N = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2.$$

Twice 2^{10} is the product of 10 2's multiplied by 2, that is, the product of 11 2's or 2^{11} , and so on. So that

$$2N = 2^{11} + 2^{10} + 2^9 + \dots + 2^4 + 2^3 + 2^2,$$

and subtracting the expression for N from that for $2N$, we have $N = 2^{11}-2$.

7. Find a corresponding expression for the number N of arrangements possible without going beyond arrangements of n signs.—We have to add up

the number of arrangements of n	signs,
„	$n-1$ signs,
„	$n-2$ „
• • • • •	• • •
„	3 „
„	2 „
„	1 „

that is

the product of n	2's	or	2^n
„	$n-1$	„	2^{n-1}
„	$n-2$	„	2^{n-2}
• • • • •	• • •	• • •	• • •
„	3	„	2^3
„	2	„	2^2
„	1	„	2

and so the number N is

$$2^n + 2^{n-1} + 2^{n-2} + \dots + 2^3 + 2^2 + 2.$$

We now double each number. Twice 2^n is twice the product of n 2's, or is the product of $n+1$ 2's or is 2^{n+1} . Twice 2^{n-1} is twice the product of $n-1$ 2's or is the product of n 2's or is 2^n , and so on. So that $2N$ is

$$2^{n+1} + 2^n + 2^{n-1} + \dots + 2^4 + 2^3 + 2^2.$$

In these two expressions 2^{n+1} and 2 occur once; but every other number (2^n , 2^{n-1} , and so on to 2^2) occurs twice. So that by subtracting we get

$$N = 2^{n+1} - 2.$$

8. *Semaphore*. Let us use for signalling an arm hinged to the top of a post so that it can be set in any of the seven

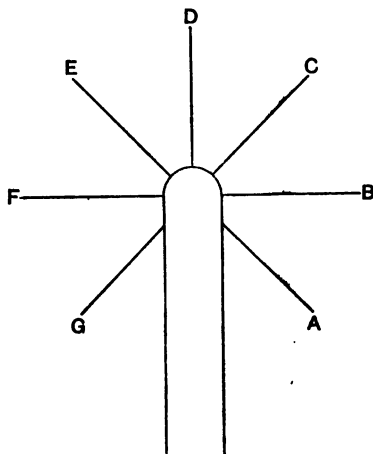


FIG. 1.

positions of Fig. 1, so that we have 7 distinct signs. By using two signs in succession how many signals can be made?—They are

AA, AB, AC, AD, AE, AF, AG,
 BA, BB
 GA, GB, GC GG,

49 in all.

With 3 signs in succession, how many signals?— $7 \times 7 \times 7$.

With 10 signs and with n signs?—The number with 10 signs is the product of 10 7's, or written shortly 7^{10} , or multiplied out about 280 million, the number with n signs is the product of n 7's or 7^n .

How many signals can be made without going beyond 3 signs for each? Without going beyond 10?—When 10

signs are allowed, we may use 10 or 9 or . . . or 1, and so get $7^{10} + 7^9 + 7^8 + 7^7 + 7^6 + 7^5 + 7^4 + 7^3 + 7^2 + 7$ signals, say N signals. Then we calculate 7 times N . 7 times 7^{10} is the product of 11 7's or is 7^{11} , and so on, so that $7N$ is $7^{11} + 7^{10} + \dots + 7^3 + 7^2$. Subtracting, we have $7N - N = 7^{11} - 7$, or $N = \frac{7^{11} - 7}{6} = \text{about } 380 \text{ million.}$ This calculation

is tedious. Later, we must try to find a shorter method.

Flag signalling.

9. A ship has a variety of flags and makes signals by arranging flags in order down a rope. With two flags A and B , how many signals are possible, both flags being used?—Two, with A above or with B above. And with 3 flags A , B , C , how many signals, using all 3 flags?—By actual trial there are 6 signals.

How would you make a 3-flag signal from a 2-flag one, say from $\begin{smallmatrix} A \\ B \end{smallmatrix}$ in which A is above?—We could put C above A , between A and B , or below B . In the same way we can make three 3-flag signals from $\begin{smallmatrix} B \\ A \end{smallmatrix}$; so that there are 6 in all.

Using a fourth flag D in addition, how many 4-flag signals are possible?—Beginning with any 3-flag signal we can insert D in the 1st, 2nd, 3rd, or 4th place, so that each 3-flag signal gives 4 4-flag signals, and the 6 3-flag signals give 24 4-flag signals.

10. Tabulate these results and extend them.—

A single flag gives	1	signal.
2 flags give	2	2-flag signals.
3 "	2×3	3-flag "
4 "	$2 \times 3 \times 4$	4-flag "
5 "	$2 \times 3 \times 4 \times 5$	5-flag "
.		
10 "	$2 \times 3 \times 4 \times \dots \times 10$	10-flag "
" "	$2 \times 3 \times 4 \times \dots \times n$	n -flag "

Roughly multiply out the number of 10-flag signals.—It is about $3\frac{1}{2}$ million.

partner the girl with the same colours. With 3 colours a, b, c , how many different rosettes can you make?—There are 3 single-colour rosettes a, b, c . The 2-colour rosettes are ab or ba, ac or ca , and bc or cb . There is one 3-colour rosette; that it contains the 3 colours may be indicated by any of the forms $abc, acb, bca, bac, cab, cba$. In all 7 rosettes.

With 5 colours a, b, c, d, e , how many 2-colour rosettes could be made?—They are $ab, ac, ad, ae; bc, bd, be; cd, ce; de$; 10 in number. If we distinguished not only colour but also the order of the two colours in a rosette, how many could be made?—The colours a and b would then give two rosettes, one with a on top, the other with b on top; and so for each pair of colours. The total number would be 20.

Find independently how many 2-colour rosettes can be made from 5 colours, both colour and order counting.

How many 3-colour rosettes can be made with 5 colours, colour alone counting?—They can be written out and seen to number 10. Or, every choice of 2 colours leaves 3 colours unchosen, so that the number of ways of choosing 3 colours is the same as the number of ways of choosing 2.

If order counts as well, how many 3-colour rosettes?—Any 3 colours will then make 6 rosettes, and the whole number will be 60. Find independently how many 3-colour rosettes can be made from 5 colours, colour and order counting.

14. How many 4-colour rosettes can be made with 10 colours? In this case direct enumeration is likely to lead to error, and is tedious besides. Can you more easily give the number when order counts as well as colour, and the number that any single selection of 4 colours gives when order counts?—When order counts, we can choose the top colour in 10 ways, the second in 9, the third in 8, and the bottom colour in 7; that is, there are $10 \times 9 \times 8 \times 7$, or 5040 rosettes. Four colours having been selected, we can from them choose the top colour in 4 ways, the second in 3, the third in 2, and the remaining colour goes at the bottom; so that the 4 colours make $4 \times 3 \times 2$, or 24 rosettes.

Can you now give the number N of 4-colour rosettes when colour alone counts?—Each of these N rosettes

would, if order counted, make 24 rosettes. So that there would be $24 \times N$ 4-colour rosettes if order counted, and we know this number to be 5040, so that $24N = 5040$, and $N = 5040/24 = 210$. Or, if we do not multiply out, we have the form

$$N = \frac{10 \times 9 \times 8 \times 7}{2 \times 3 \times 4}.$$

15. With 100 colours, how many rosettes of 1, 2, 3, and 4 colours can be made, colour alone counting?—The numbers are respectively

$$100; \frac{100 \times 99}{2}; \frac{100 \times 99 \times 98}{2 \times 3}; \frac{100 \times 99 \times 98 \times 97}{2 \times 3 \times 4};$$

or approximately 100; 4950; 162,000; 3,920,000.

With n colours, how many rosettes of 1, 2, 3, and 4 colours can be made, colour alone counting?—The numbers are

$$n, \frac{n(n-1)}{2}, \frac{n(n-1)(n-2)}{2 \times 3}, \frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4}.$$

Any selection of 4 colours or other objects from among n different colours or other objects is called in mathematics a combination of n things 4 at a time. In general, a selection of p things from among n things all different is a combination of n things p at a time.

Binomial Theorem.

16. To multiply 83 by 7 we add together 7 times 80 and 7 times 3. No matter what values a b c have, a times $b+c$ is the sum of a times b and a times c , or in mathematical form $a \times (b+c) = a \times b + a \times c$, or $a(b+c) = ab + ac$. In this way multiply $x+a$ and $x+b$ together.—The product is $xx+ax+bx+ab$. Multiply this product by

$x+c$ and put first the term containing x three times, then the terms containing x twice, then those containing x once, and last of all the term without x .—The new product is $xxx+axx+bx+cx+abx+bcx+acx+abc$.

The terms containing x twice are often grouped together and written $(a+b+c)xx$, or with the index notation $(a+b+c)x^2$. Group the terms containing x once in the same way, and write the whole product, using the index notation.—It is $x^3 + (a+b+c)x^2 +$

$(ab + bc + ca)x + abc$. In this expression the number $a + b + c$ that multiplies x^2 is called the coefficient of x^2 , and $ab + bc + ca$ is called the coefficient of x . Multiply this expression by $x + d$ and group the product according to powers of x .—It is $x^4 + (a + b + c + d)x^3 + (ab + bc + cd + da + ac + bd)x^2 + (abc + bcd + cda + dab)x + abcd$.

17. In this product what is the relation between each coefficient and the numbers $a b c d$?—The coefficient of x^3 is the result of selecting one of the numbers at a time and adding the selections. In the coefficient of x^2 occurs every possible combination of the numbers $a b c d$, taken 2 at a time; the coefficient is the sum of all possible products 2 at a time. The coefficient of x is the sum of all possible products of $a b c d$, taken 3 at a time. The term that does not contain x is the only possible product of the four numbers all together.

18. When the product $(x + a)(x + b)(x + c)(x + d)$ is multiplied out, how many of the quantities $x a b c d$ occur in each term?—By looking at the multiplying out we see that every term contains one letter from each factor, and that every possible combination of one letter from each factor appears.

How many terms then are there altogether? And how many groups when we group together all the terms that contain x the same number of times?—The number of ways of choosing one letter from each factor is $2 \times 2 \times 2 \times 2$, or 16; so this is the number of terms reckoned this way. As to groups, a group may contain x^4 or x^3 or x^2 or x , or not contain x at all; so there are five groups.

How is the group containing x^3 made up?—It takes x from three factors and a different letter from the remaining factor.

And the group containing x^2 ?—It takes x from two factors and other letters from the remaining two. It will contain as multipliers of x^2 every possible combination, 2 at a time, of the numbers $a b c d$. How are the remaining groups made up?

19. Suppose for a moment that $a b c d$ all have the value 1. How will you write $(x + 1)(x + 1)(x + 1)(x + 1)$ more shortly? And what are the groups of the multiplied-out form?—With the index notation the expression is written $(x + 1)^4$. The term containing x four times occurs

once only, by taking x from each factor; it is simply x^4 . The term not containing x occurs once only by taking 1 from each factor; it is 1. Terms containing x three times occur by taking 1 from any one factor and x from the other three; four such terms occur, so the group is $4x^3$. Terms containing x twice occur by taking 1 from any two of the factors and x from the remaining factors; the number of such terms is the number of ways of choosing 2 factors from the 4 factors, that is $4 \times 3 \div 2$, or 6; so the group is $6x^2$. Terms containing x once occur by taking x from any one factor and 1 from the remainder; this one factor can be chosen in 4 ways, so the group is $4x$.

Verify these results by substituting 1 for a, b, c, d in the expression found for $(x+a)(x+b)(x+c)(x+d)$, and also by simply multiplying out $(x+1)^4$.

20. Give the various groups of terms when $x+a, x+b, x+c, x+d, x+e, x+f, x+g$ are multiplied together. How many terms are there in each group?

Also find the number of terms by multiplying out $(x+1)^7$, and check the result by actually multiplying together $x+a, x+b, \&c.$

21.* When n factors $x+a, x+b, \dots, x+t$ are multiplied together, how many terms are there in the groups containing $x^n, x^{n-1}, x^{n-2}, x^{n-3}, x^{n-4}$?—The numbers of terms are

$$1, \quad n, \quad \frac{n(n-1)}{2}, \quad \frac{n(n-1)(n-2)}{2 \times 3}, \quad \frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4}.$$

When $(x+1)^n$ is multiplied out, write down the first five groups.—They are

$$x^n, \quad nx^{n-1}, \quad \frac{n(n-1)}{2} x^{n-2}, \quad \frac{n(n-1)(n-2)}{2 \times 3} x^{n-3}, \\ \frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4} x^{n-4}.$$

The result that

$$(x+1)^n = x^n + nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} + \dots$$

* The remaining articles of this chapter should be postponed till they are wanted in article 4 of chapter XII.

is known as the binomial theorem, and the series $x^n + nx^{n-1} + \dots$ is called the binomial series.

22. When the n factors $x+a, \dots, x+t$ are multiplied together, how many terms are there that do not contain x , that contain x once, twice, &c. ? When $(x+1)^n$ is multiplied out, what are the last few groups ?—They are

$$\frac{n(n-1)(n-2)}{2 \times 3} x^3, \quad \frac{n(n-1)}{2} x^2, \quad nx, \quad 1.$$

The binomial theorem can now be more completely stated thus :—

$$(x+1)^n = x^n + nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} + \dots + \frac{n(n-1)}{2} x^2 + nx + 1$$

or, written according to ascending instead of descending powers of x ,

$$(x+1)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{2 \times 3} x^3 + \dots + nx^{n-1} + x^n.$$

Interest on a loan.

23. A sum of money of £ a is lent at interest at b per cent. per year, the interest not being paid, but being added at the end of every year to the loan and so itself bearing interest in succeeding years. What is the interest for the first year ?—It is, in pounds, $\frac{a \times b}{100}$. And the sum that bears interest during the second year ?—It is

$$a + \frac{ab}{100}, \text{ or } a\left(1 + \frac{b}{100}\right).$$

What is the interest earned during the second year, and the sum that bears interest in the third year ?—The interest is $\frac{b}{100}$ times $a\left(1 + \frac{b}{100}\right)$, and the sum that bears interest in the third year is $a\left(1 + \frac{b}{100}\right)$ plus $\frac{b}{100}$ times $a\left(1 + \frac{b}{100}\right)$, that is, $a\left(1 + \frac{b}{100}\right) \times \left(1 + \frac{b}{100}\right)$. What is the sum

that bears interest in the fourth year, in the fifth year, and in the tenth year?—The sum in the fourth year is

$$a\left(1+\frac{b}{100}\right)\left(1+\frac{b}{100}\right)\left(1+\frac{b}{100}\right), \text{ or } a\left(1+\frac{b}{100}\right)^3,$$

in the fifth $a\left(1+\frac{b}{100}\right)^4$, and in the tenth $a\left(1+\frac{b}{100}\right)^9$.

24. And what does the whole amount to at the end of the tenth year, original loan and accumulated interest? Calculate this amount when the sum lent is £740 and the rate of interest is 3 per cent., first by direct calculation, and then by the help of the binomial theorem.—The number of pounds is $740 \times \left(1+\frac{3}{100}\right)^{10}$. We multiply £740 ten times by 1·03 and find £994 10s. 0d. The binomial theorem gives:—

$$\left(\frac{3}{100}+1\right)^{10} = \left(\frac{3}{100}\right)^{10} + 10 \times \left(\frac{3}{100}\right)^9 + \frac{10 \times 9}{2} \left(\frac{3}{100}\right)^8 + \dots$$

or, in ascending powers of $\frac{3}{100}$,

$$= 1 + 10 \times \frac{3}{100} + \frac{10 \times 9}{2} \left(\frac{3}{100}\right)^2 + \dots$$

Now $\left(\frac{3}{100}\right)^{10}$ is only 0·0000000000000006, and the terms beside it are not much bigger. Let us therefore begin at the other end, that is, with the series in ascending powers. We calculate the successive terms to five decimal places and place them in a column.

1
0·3
0·0405
0·00824
0·00017
0·00001

The successive terms become smaller and smaller till the seventh and following terms are too small to affect the

fifth decimal place. The sum of the six terms is 1·84892, and the accumulated value of the loan is $740 \times 1\cdot84892$ pounds, or £994 10s. 0d.

In this case it is doubtful whether the use of the binomial series saves any time over direct calculation. Presently we shall have calculations for which there is no comparison between the methods.

EXERCISES.

2. A railway has 34 stations. How many kinds of third class single tickets does it take to supply the system?

3. A school has 87 girls, and each of them sends a Christmas card to every other. How many do they send?

4. The floor of a conservatory is to be paved with red, white, and black tiles according to the design indicated in Fig. 2. If the length and breadth of the floor are respectively 40 and 20 times the length of the diagonal of a square tile, find the number of square tiles of each colour that will be required, ignoring the smaller tiles.

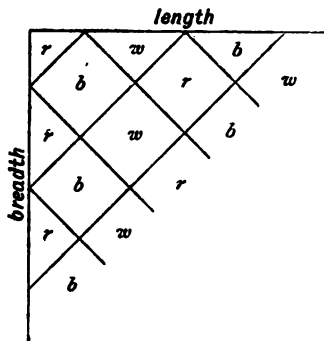


FIG. 2.

5. In the game of dominoes a number of blocks are used. Each block is marked off into two parts, and on each of these parts a number of pips, between 0 and 6 inclusive, are marked. These are arranged in all possible ways, and the pips on the two parts of a block may be the same or different; and no two blocks are alike. How many blocks are there?

6. A publisher uses sheets of paper 24 inches by 10 inches to make a book each page of which measures 8 inches by 5 inches. He folds each sheet of paper so that it makes 6 leaves or 12 pages. Sketch a sheet of paper, back and front, mark the creases and write on the several pages the numbers 1, 2, . . . 12, so that when the sheet forms part of the book these numbers will appear in proper order right side up. Mark corresponding corners of the back and front views of the sheet with the letters *A*, *B*, *C*, *D*.

7. Nine cards numbered 1 to 9 are arranged in three heaps as in Fig. 3. The bottom card of heap 1 is taken away and placed on

top of heap 2; then the card now at the bottom of heap 1 is taken and put on heap 3. Similarly, the bottom card of heap 2 is placed on top of heap 3; and the card left at the bottom of heap 2 put on

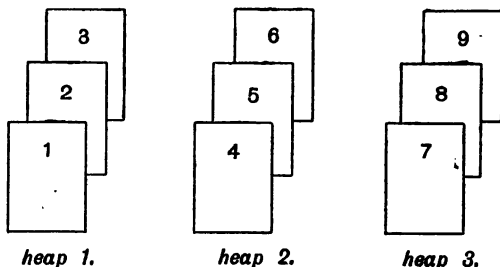


FIG. 3.

heap 1. Similarly, the card at the bottom of heap 3 is placed on heap 1, and the card so left at the bottom on heap 2. Sketch the cards in their position at this stage.

When this operation is repeated, name the positions taken up in succession by card No. 1. After how many repetitions does this card return to its original position? Are the other cards then in their original positions? How do you know?

8. A marble statue is 5 feet 6 inches high. A statuette which is an exact copy of the statue on a smaller scale and in plaster weighs $\frac{1}{8}$ lb. The statuette is 9 inches high and is half the weight it would have been in marble. What is the weight of the statue, to the nearest hundredweight? Note that:—

$$\frac{\text{volume of statue}}{(\text{height of statue})^3} = \frac{\text{volume of copy}}{(\text{height of copy})^3}.$$

9. A wooden ball 25 centimetres in diameter weighs 5.9 kilograms. Find, to the nearest hundredth of a gram, the weight of a cubic centimetre of the wood. The volume of a ball of diameter d is $\frac{1}{6}\pi d^3$, where $\pi = 3.14$.

10. The number of cubic feet of timber in an oak log, which tapers uniformly, may be calculated from the formula $0.2618h \times (a^2 + ab + b^2)$, where a and b are the numbers of feet in the diameters of the ends of the log, and h is the number of feet in the length of the log. Find what is the weight of such a log of oak whose length is 15 feet, and the diameters of whose ends are 3 feet and 2 feet respectively, assuming that a cubic foot of the oak weighs 50 lb. Give your answer to the nearest cwt.

If you assumed the diameter to be 2 feet 6 inches throughout,

how much per cent. (to the nearest integer) would the result be wrong?

11. Find the value of

$$1 + mn + \frac{m(m-1)}{2}n^2 + \frac{m(m-1)(m-2)}{6}n^3$$

(1) when $m = 2$ and $n = 3$, and also (2) when $m = 24$ and $n = \frac{1}{2}$.

12. The volume and surface of a sphere are respectively $\frac{4}{3}\pi r^3$ and $4\pi r^2$, r being its radius. A sphere S , whose radius is 10 centimetres long, is covered with a substance which is everywhere 1 millimetre thick. Show that the volume of the substance is nearly equal to the surface of S multiplied by the thickness of the substance, and find the percentage error involved in assuming this to be the case.

13. The present value of an annuity A for n years reckoned at r per cent. compound interest is equal to $\frac{R^n - 1}{R^n(R - 1)}A$, where $R = 1 + \frac{r}{100}$. Find the present value of an annuity of £50 for 15 years at $2\frac{1}{2}$ per cent.

14. In Fig. 4 DC is the graph of $y = ax + b$, and $OB = x_1$. Express the area of $OB CD$ in terms of x_1 , a and b .

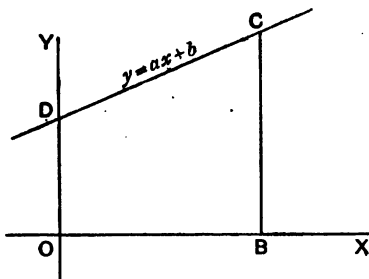


FIG. 4.

If such a line as DC is drawn, and it is found that for all values of x_1 the area of $OB CD$ is $\frac{1}{2}px_1^2 + cx_1$, what must be the equation of DC ?

15. By drawing find the sines of 28° and 74° . Make (when possible) angles whose sines are 0.21, 0.42, 0.84, 1.68, and measure them with your protractor.

Check your results by referring to a book of tables.

16. Draw two straight lines AOB and COD crossing at right angles. On tracing-paper draw a line PQ 8 cm. long, and on PQ mark R 1 cm. from P . Move the tracing-paper so that P always lies on AB and Q on CD . Prick through the point R in various positions, and so draw the path of R .

Repeat, taking R in succession at distances of 2, 4, 6, 7 cm. from P .

In the case when R is midway between P and Q , discuss by general reasoning the nature of the path of R .

17. Draw a straight line and a curve between two points A and B . Erect perpendiculars at intervals of 1 centimetre along the straight line AB . Join the points at which successive perpendiculars meet the curve. Estimate approximately the area of the figure by assuming it to be equal to the polygon so made.

18. Water runs at $4\frac{1}{2}$ miles an hour along a pipe the cross-section of which is 3 square feet. How many gallons (to the nearest million) does it deliver in 24 hours? A cubic foot is 6.23 gallons.

19. On tracing-paper draw lines OX and OY , meeting at an angle of 80° , making OX 10 cm. and OY 7.5 cm. long; joining these lines draw others X_1Y_1 , X_2Y_2 , X_3Y_3 , . . . parallel to XY and 0.4 cm. apart. Draw a semi-circle bounded by a diameter, and mark T the centre and S any point within the boundary. Place the tracing-paper on this semi-circle with O above the point S , and pin it at this point. Now rotate the tracing, and as any point on OX (say X_1) comes over the boundary of the semi-circle, prick through Y_1 the corresponding point on OY ; when X_2 is on the boundary prick through Y_2 ; and so on. Draw the complete locus of Y as X traces out the given figure.

What would the locus of Y be if O were pivoted at T ?

CHAPTER XI

ON CARRYING WATER

1. A MAN has to take a bucket from a cottage C , fill it at some point P of a ditch D , and carry it to a trough T . Taking the distances as shown in Fig. 1, find, by drawing

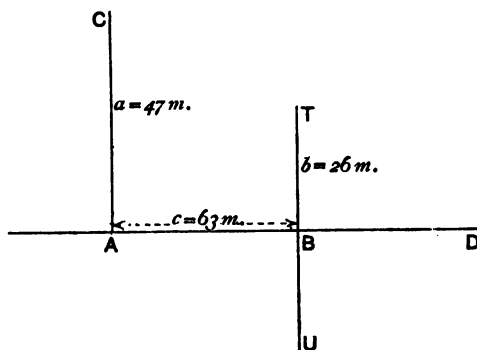


FIG. 1.

to a suitable scale, the length of the journey for different positions of P , say at distances of 5, 10, 15, . . . metres from A ; and show your results in a table.

At every point P draw a $\perp$ to the ditch, of a length that represents (on the scale chosen) the length of the journey for that position of P . By drawing a curve through the ends of these $\perp$ s, make a graph showing the length of the journey for any position of P . From your graph find the position of P that gives the shortest journey.

2. Produce TB to U so that $BU = BT$, that is, take U the image of T in the line of the ditch. Suppose the man to go from C to U , crossing the ditch at some point P ,

and going straight from C to P and from P to U . Give a graph showing the length of the journey for different positions of P . From the graph find which journey is the shortest. Can you by any better method than the graph decide which is the shortest journey from C to U ?—One path is straight. It is therefore the shortest.

Note for the teacher. It is our constant experience that the straight path is the shortest. Whether this connexion between straightness and shortness is 'in the nature of things' does not concern us, as it is not suited for discussion by the young.

Can you now find the shortest route from C to T by a better method than the graph?— P being any point on the ditch, folding our figure about the ditch brings T and U together, and brings PT and PU together, so that any route CPT is equal to the route CPU . Therefore, for the shortest route the bucket must be filled at Q , where the straight path CU cuts the ditch.

3. So far, P has been assumed to lie on the stretch AB of the ditch. How does the route CRT of Fig. 2 compare with the route CAT ?—Observation shows that when P lies on AS and moves towards S , each piece, CP and TP , of the

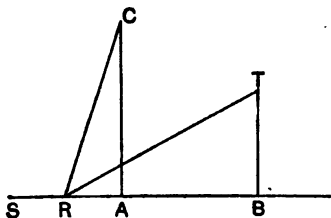


FIG. 2.

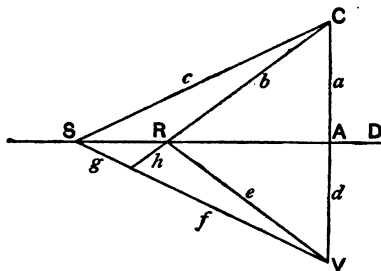


FIG. 3.

route continually increases.

route continually increases. Give a graph showing the distance CR as a function of the distance AR . Which point of D is nearest to C ? Can you prove it?— V being the image of C in the line of the ditch (Fig. 8), the straight

route CAV is less than the crooked route CRV , and CA , half the straight route, is less than CR , half the crooked one.

Can you show that, if S lies further than R from A , CS is greater than CR ? Name the distances as in Fig. 3, and write down all the relations you can among a, b, c, d, e, f, g, h .—The relations immediately useful are $c+g > b+h$, and $h+f > e$, which give $c+g+f > b+e$, or $c > b$, since $c = g+f$ and $b = e$. State the conclusion formally.

—When a perpendicular stands on another line, a point travelling along this line away from the foot of the perpendicular is constantly increasing its distance from the head of the perpendicular also.

4. Consider again a triangle ODE , in which $\angle D$ is right. Make $\angle EDF = \angle OED$, as in Fig. 4. Name the lengths as in the figure, and give all the relations you can among them. Do these help towards a proof that $OD < OE$?—Since $\angle FDE = \angle FED$, we know that $p = s$. And $q+s > r$, so that $q+p > r$, that is, $OE > OD$.

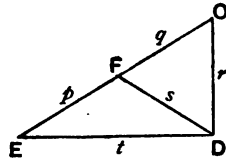


FIG. 4.

Generalize this result.—For the proof it does not matter that ODE is a right angle, it matters only that $\angle ODE$ is greater than $\angle OED$. So that any triangle ODE that has $\angle ODE > \angle OED$ has also $OE > OD$.

Discuss the converse proposition. If you know OE to be $> OD$, can you tell anything of the angles OED and ODE ? In the former case you cut from the greater angle a piece equal to the smaller. Try now cutting from the greater side a piece equal to the smaller side, and write down all the relations you can among the angles.—Cut off OG equal to OD , and name the angles as in Fig. 5. The angle q , being the exterior angle of the triangle DEG , is equal to the interior angles s and t together, and is therefore greater than t . Also the triangle ODG being isosceles, q is equal

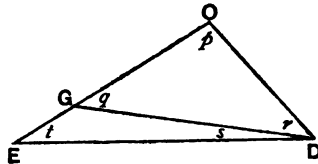


FIG. 5.

to r . We know then that t is less than r which is a part of $\angle ODE$, therefore $\angle OED$ is less than $\angle ODE$.

State these results formally.—The greater angle of a triangle has the greater side opposite, and the greater side has the greater angle opposite. Or, since the reasoning applies to any two of the three sides or angles, the order of size of the angles of a triangle is the same as the order of size of the opposite sides.

5. Draw again the ditch D , the cottage C , the trough T , the image U of the trough, the point Q that gives the shortest route, and let the line joining C to T meet the ditch in Z , as in Fig. 6. Name all cases of equal ratios you can find

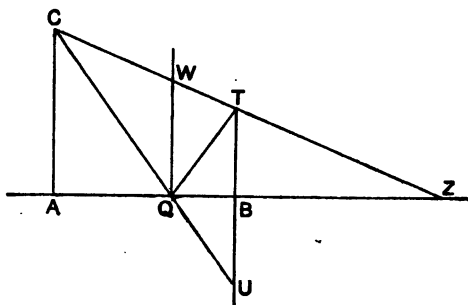


FIG. 6.

by measuring lines and calculating ratios. In particular, give all the ratios that are equal to $\frac{QC}{QT}$. Draw $QW \perp$ to AB and see if it gives any more ratios equal to $\frac{QC}{QT}$. Confirm your conclusions by general reasoning.— QU being $= QT$, the ratio $\frac{QC}{QU}$ is $= \frac{QC}{QT}$. TBU being $\parallel CA$, the Δ s BUQ and ACQ are similar, and $\frac{QC}{QU} = \frac{AC}{BU} = \frac{QA}{QB}$; and $\frac{AC}{BU} = \frac{AC}{BT}$. That TBU is $\parallel CA$ gives Δ s CAZ and TBZ similar, and $\frac{CA}{TB} = \frac{CZ}{TZ} = \frac{AZ}{BZ}$. Since QW is $\parallel TBU$,

$$\frac{CQ}{QU} = \frac{CW}{WT}.$$

Compare the angles TQB and UQB , and the angles CQW and TQW . Keep only enough of the figure to show the equal ratios $\frac{QC}{QT}$, $\frac{WC}{WT}$, $\frac{ZC}{ZT}$, and remove the rest. What conditions with regard to this reduced figure must hold to make these three ratios equal?—It is enough to know that QW bisects the angle CQT , and that QZ is

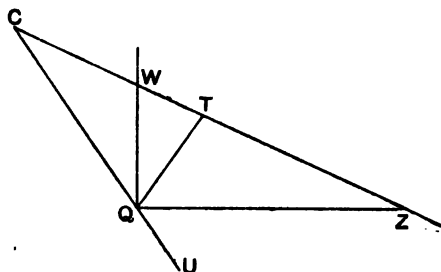


FIG. 7.

1. QW or bisects the angle TQU (Fig. 7), for this enables us to restore the original figure. Show how to restore the original figure.

6. Let us call Q the vertex of the triangle QCT . Then QW is the bisector of the vertical angle of the triangle. The angle TQU , made by producing a side CQ , is called the external vertical angle, and QZ is the bisector of the external vertical angle. The point W (in ordinary English) divides the base CT into two parts, namely, from W to C , and from W to T . By an extension of the language the point Z is also said to 'divide' the base CT into two parts, namely, the part from Z to C and the part from Z to T . By means of these terms state the result concisely.—The bisector of the vertical angle of a triangle divides the base into two parts that are in the same ratio as the two sides; the bisector of the external vertical angle divides the base into two parts that are also in the same ratio.

7. Consider the propositions converse to those just stated. State them, and see whether they are true.—If a point W

(Fig. 8) divides the base CT of a triangle CQT internally so that $\frac{CW}{WT} = \frac{CQ}{QT}$, then WQ bisects the vertical angle CQT . Draw $TU \parallel QW$, and meeting CQ in U . Then

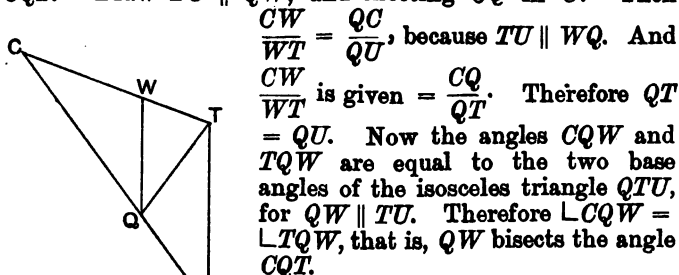


FIG. 8.

$\frac{CW}{WT} = \frac{CQ}{QT}$, because $TU \parallel WQ$. And $\frac{CW}{WT}$ is given $= \frac{CQ}{QT}$. Therefore $QT = QU$. Now the angles CQW and TQW are equal to the two base angles of the isosceles triangle QTU , for $QW \parallel TU$. Therefore $\angle CQW = \angle TQW$, that is, QW bisects the angle CQT .

Having given that a point Z divides the base CT of $\triangle CQT$ externally so

that $\frac{ZC}{ZT} = \frac{QC}{QT}$, draw $TY \parallel ZQ$ to

meet CQ in Y , and use a similar proof to show that QZ bisects the external vertical angle of the triangle.

8. Return to the original problem of carrying water by the shortest route. In what ways can you specify the point of the ditch at which the bucket should be filled?—

(1) It is the point where the line from C to the image of T crosses the ditch. (2) It is the point Q such that the two parts QC and QT of the path are equally inclined to the ditch. (3) It is the point Q that divides AB internally in the same ratio that the line CT divides it externally.

9. From the data in Fig. 1 calculate the distances of Q and Z (Fig. 6) from A and B .—All lengths being in

metres, we have $\frac{ZA}{ZB} = \frac{CA}{TB} = \frac{47}{26}$ and $\frac{ZA}{ZB} = \frac{ZB + BA}{ZB}$

$= 1 + \frac{BA}{ZB} = 1 + \frac{63}{ZB}$, so that $\frac{63}{ZB} = \frac{47}{26} - 1 = \frac{21}{26}$; or ZB

$= \frac{63 \times 26}{21} = 78$. Hence $ZA = 78 + 63 = 141$. Again, in

the same way $\frac{AQ}{QB} = \frac{CA}{UB}$, that is, $\frac{63 - QB}{QB} = \frac{47}{26}$; and

solving this equation we find $QB = 22.4$, and therefore $QA = 40.6$.

Check these results by measuring a figure drawn to scale. Measure also the angles made by QC , QT , and CZ with AB . Measure CQ , TQ , CZ , TZ , and express (from your measurements only and the original data) $\frac{CQ}{QT}$, $\frac{CZ}{TZ}$, $\frac{CA}{TB}$ as decimals.

What name is given to $\frac{CA}{CQ}$ in relation to the angle CQA ? To $\frac{TB}{TQ}$ in relation to $\angle TQB$? To $\frac{CA}{CZ}$ and $\frac{TB}{TZ}$ in relation to $\angle CZA$? Calculate all these as decimals, use a table of sines to find from them the angles CQA , TQB , CZA , and compare with the measured values of the angles.

10. Draw at right angles two lines QX and QY (Fig. 9). Draw two lines QC and QT , making with QY the same

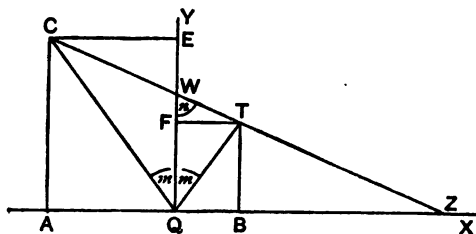


FIG. 9.

acute angle. Suppose this angle to contain m degrees, suppose QC to contain r cm. and QT to contain s cm. Draw CA and $TB \perp$ to QX , mark all the angles that are equal to m , and express any lengths you can in terms of r , s , m .

—Since $\frac{QA}{QC} = \sin m$, we see that $QA = QC \times \sin m$, that is, $= r \times \sin m$, or, dropping the sign of multiplication, $QA = r \sin m$. Also $QB = s \sin m$.

We have sometimes selected one side of a triangle and called it the base and no longer called it a side. So here the

longest side of a right-angled Δ will be called the hypotenuse and no longer a side. In a right-angled triangle you know that the sine of an acute angle means the ratio of the side opposite the angle to the hypotenuse. Another useful term is cosine. The cosine of an angle is the ratio of the side beside the angle to the hypotenuse. Thus in ΔCQA the

cosine of m is $\frac{CA}{CQ}$; it is written 'cos m '. Use this term to give additional lengths on Fig. 9 in terms of r , s , m .—Cos $m = \frac{CA}{CQ} = \frac{TB}{TQ}$, so that $CA = r \cos m$ and $TB = s \cos m$.

11. Join CT and let it meet QX in Z and QY in W . Draw CE and $TF \perp$ to QY . Suppose the acute angle between QY and CT an angle of n degrees. Mark all angles of n degrees on the figure, and express any additional lengths you can in terms of r , s , m , n .—All the ratios

$\frac{ZA}{ZC}$, $\frac{ZB}{ZT}$, $\frac{CE}{CW}$, $\frac{TF}{TW}$, $\frac{ZQ}{ZW}$ are equal to $\sin n$. The third of these ratios gives $CW \sin n = CE = AQ = r \sin m$, so that $CW = \frac{r \sin m}{\sin n}$; and the fourth ratio gives $TW \sin n =$

$TF = BQ = s \sin m$, so that $TW = \frac{s \sin m}{\sin n}$. All the ratios

$\frac{CA}{CZ}$, $\frac{TB}{TZ}$, $\frac{WE}{WC}$, $\frac{WF}{WT}$, $\frac{WQ}{WZ}$ are equal to $\cos n$. They give

$CZ = \frac{CA}{\cos n} = \frac{r \cos m}{\cos n}$; $TZ = \frac{TB}{\cos n} = \frac{s \cos m}{\cos n}$; and other relations.

12. From the values found for CW , TW , CZ , TZ deduce the ratios $\frac{CW}{TW}$ and $\frac{CZ}{TZ}$.— $\frac{CW}{TW}$ is equal to

$\frac{r \sin m}{\sin n} \div \frac{s \sin m}{\sin n}$ or $\frac{r}{s}$. $\frac{CZ}{TZ} = \frac{r \cos m}{\cos n} \div \frac{s \cos m}{\cos n}$ or $\frac{r}{s}$.

These results are a further proof that the two bisectors of the vertical angle of ΔQCT divide the base in the ratio of the sides QC and QT (or r and s).

Ex. 1. Give m the value 41° , make $r = 74$, $s = 85$, draw the figure, measure n and calculate the lengths CW , TW , CZ , TZ . Compare these with the measured lengths. The positions of C and T may be given by their distances from QX and QY ; calculate these and compare with their measured lengths.

13. One more ratio connected with a right-angled triangle must be named. The ratio of the side opposite an acute angle to the side beside the angle is called the tangent of the angle. (Here again the term 'side' does not apply to the hypotenuse.)

Thus $\frac{QA}{CA}$ is the tangent of m , and is written ' $\tan m$ '.

The values of m , $\sin m$, $\cos m$, $\tan m$ are all given when we know the value of m or of one of the ratios, so that these ratios are related.

Give one relation between $\sin m$, $\cos m$, and $\tan m$.— $\sin m = \frac{AQ}{QC}$, $\cos m = \frac{CA}{QC}$, so that $\frac{\sin m}{\cos m} = \frac{QA}{QC} \div \frac{CA}{QC} = \frac{QA}{CA} = \tan m$.

By expressing CT (or other length of Fig. 9) in different ways, find a relation between m and n .— $CW = \frac{r \sin m}{\sin n}$

and $TW = \frac{s \sin m}{\sin n}$, so that $CT = \frac{(r+s) \sin m}{\sin n}$. Also

$CZ = \frac{r \cos m}{\cos n}$ and $TZ = \frac{s \cos m}{\cos n}$, so that $CT = \frac{(r-s) \cos m}{\cos n}$.

Hence, $\frac{(r+s) \sin m}{\sin n} = \frac{(r-s) \cos m}{\cos n}$, or $\frac{(r+s) \sin m}{\cos m} = \frac{(r-s) \sin n}{\cos n}$, that is, $(r+s) \tan m = (r-s) \tan n$.

By means of this relation and mathematical tables calculate n when $r = 74$, $s = 85$, $m = 41$. Hence, calculate the lengths CW , TW , CZ , TZ , and the ratios $\frac{r}{s}$, $\frac{CW}{TW}$, $\frac{CZ}{TZ}$.

EXERCISES.

2. Draw an angle whose sine is 0.4, and find the value of the angle, of its cosine, and of its tangent.

3. An angle has a cosine of 0.6. Find the angle, its sine, and its tangent.

4. The tangent of an angle is 4. What are the angle, its sine, and its cosine?

5. Can the cosine of an angle be equal to 2? What values can the cosine have? And the sine? And the tangent?

6. Having given a line AB and two points C and D , find a point E in AB such that EC and ED are equally inclined to AB .

7. Having given a piece AB of a straight line, show how to divide it internally or externally in a given ratio. Is the method of calculation or of drawing more convenient when the ratio is given as a number? Which is more convenient when the ratio is given as that of two lines shown?

8. By drawing and measuring find the values of the sine, cosine, and tangent of angles of 10, 20, 30, 40, 50, 60, 70, 80 degrees.

For each of these angles verify that the tangent is the quotient of the sine by the cosine.

Draw the graphs of the sine, cosine, and tangent of an angle from 0 to 90 degrees.

9. With your protractor make an angle of 44° . By drawing find the values of the sine and tangent of this angle.

Tabulate beside these the values given in the tables and account for the differences in the results.

10. A picture is hung with the top horizontal by a cord 2 feet long fastened to the two top corners and passing over a nail, each part of the string making an angle of 60° with the top of the picture. How much will the picture be raised by shortening the cord by 6 inches and adjusting the picture so that the top is still horizontal?

11. Find, by drawing and by calculation, the altitude of the sun when a man casts a shadow twice as long as himself. (The altitude is the angle between the man's shadow and the direction of the sun's rays.)

12. In a survey it was necessary to continue a straight line XO past an obstacle which could not be seen over. A line OY was measured at right angles to XO , and from the point Y lines YP and YQ were drawn which cleared the obstacle: the angles OYP , OYQ were 36° and 53° respectively, the distance OY was 150 yards. What lengths were set off along YP and YQ so that PQ was in the same straight line with XO ?

13. A survey line is run 3.47 chains 18° north of east from A to B , 4.50 chains due east from B to C , and 3.16 chains 70° north of east from C to D . Find, by calculation, how far north and east each straight piece goes, and (to three significant figures) how far north and east D is of A .

Check your results by a drawing to scale.

14. Draw a triangle ABC , having $AB = 14$ cm., $AC = 11$ cm., $\angle BAC = 90^\circ$. Draw AD bisecting the angle BAC and meeting the base in D . Draw DE parallel to AC to meet AB

in E , and DF parallel to AB to meet AC in F . Calculate to three significant figures the ratio $\frac{AB^2}{AC^2}$; and measure BE , ED , CF , FD , and so find, to three significant figures, the ratio $\frac{\text{area } BED}{\text{area } DFC}$.

Prove that if the vertical angle of a triangle is bisected by a straight line which cuts the base of the triangle, then the base is divided into segments which are in the ratio of the sides containing the vertical angle.

The values found for the two ratios in the first part of the question suggest a simple relation that might hold between them if the drawing and measuring could be carried out with absolute accuracy. State this relation, and prove that for perfect figures it would hold for all lengths of the sides AB and AC .

15. Draw the graph on ruled paper of $\tan \frac{1}{2}x$ from $x = 0^\circ$ to $x = 360^\circ$.

16. Draw two straight lines OX and OY at right angles, OX to the right, OY up, and find a point P 4 inches from OX and 1 inch from OY . Now imagine P to remain fixed while YOX is revolved, counter-clockwise, about the point O . Determine, both by drawing and calculation, (i) the distance of P from OX when OX has turned through 60° ; (ii) what further amount of revolution would make OX pass through P .

17. SM is a road and F a point six miles from M the nearest point of the road. The distance SM is also six miles. A man who can run a miles an hour on the road and b miles an hour off it, wants to get to F . How long will he take from S to F if he leaves the road at P , x miles from M ?

When $a = 10$ and $b = 5$, find by a graph, or otherwise, what must be the value of x (to the nearest tenth) so that the man may reach F in the shortest possible time.

18. The sheet of paper $ABCD$ shown in Fig. 10 is creased along PQ so that C falls on AB (at R). If the dimensions are as indicated, and if the angle θ is 26° , find the length of PQ . You may use the tables or make a construction to scale.

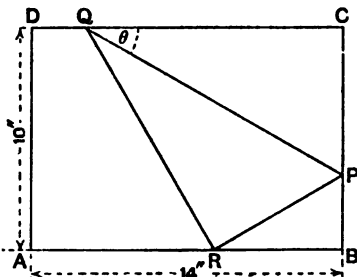


FIG. 10.

19. A thin rod, 25 inches long, is pivoted at one end, and swings, like a pendulum, in a vertical plane, passing in front of a horizontal scale fixed 20 inches below the pivot. The scale is to

be graduated so that the angle of swing of the rod to right or left of the vertical position, may be read directly on it. Where must the marks corresponding to swings of 10° , 20° and 30° on each side be placed, and what is the greatest swing we can read on the scale?

20. There are four stations on a line of railway. How many different kinds of single third-class tickets must be supplied for use on the line?

21. The area of the Trent basin is 4,082 square miles, and the annual rainfall 31 inches. The river discharges 220,000 cubic feet per minute. What percentage (to the nearest integer) of the rain that falls is discharged by the river?

22. A recent Blue Book gives the weekly expenses in summer of a number of households as follows:—The average weekly family expenditure in

	s.	d.
261 cases was	21	4
289 ,, 	27	0
416 ,, 	31	11
382 ,, 	36	6
596 ,, 	52	0

How many cases are included, what is their total expense for the week, and what is (to the nearest penny) the average for the whole?

23. How many shot, $\frac{1}{8}$ -inch in diameter, will go to a pound if a cubic foot of lead weighs between 708 and 712 lb.? [The volume of a sphere is $\frac{4}{3}\pi \times (\text{radius})^3$.]

24. An air-pump with a cylinder of volume A is used to exhaust a vessel of volume B . After n strokes of the pump the pressure of the air in the vessel is $\left(\frac{B}{A+B}\right)^n \times p$ where p is the atmospheric pressure. Taking p as 30 inches of mercury, A as 17 cubic inches, and B 130 cubic inches, find the pressure in inches of mercury after 15 strokes and after 50 strokes.

25. Draw two lines QP and QR meeting at an angle. Find whether every point on the line that bisects $\angle PQR$ is equidistant from QP and QR ; and whether every point equidistant from QP and QR lies on the bisector of $\angle PQR$.

CHAPTER XII

ON SHORTENING CALCULATIONS THE SLIDE RULE

1. RECALL the meaning of the terms index and power (chap. IX, art. 15). Calculate the powers of 2 up to the index 15 and exhibit them in a table.—The product of 2 2's is 4, of 3 2's 8, of 4 2's 16, and so on. Call the index x and the power y , so that $y = 2^x$, and the table is as follows:—

x	y	x	y
1	2	9	512
2	4	10	1,024
3	8	11	2,048
4	16	12	4,096
5	32	13	8,192
6	64	14	16,384
7	128	15	32,768
8	256		

This table enables us to perform some multiplications with little labour. Suppose, for instance, we want to multiply 128×16 . The table shows that 128 is 2^7 or 7 2's multiplied together, and that 16 is 2^4 or 4 2's multiplied together. Thus 128×16 is $7+4$ or 11 2's all multiplied together, or 2^{11} . The table shows that 2^{11} is 2,048, which is therefore equal to 128×16 .

In the same way find the products 256×128 and 512×32 .

Can you use the table to find the quotient $16,384 \div 512$?

—The table shows 16,384 to be the product of 14 2's, and

512 to be the product of 9 2's. So we have to divide the product of 14 2's by 9 2's, which leaves the product of 5 2's, and the table gives 32 as the product of 5 2's. So that $16,384 \div 512 = 32$.

2. But we must deal with many numbers that do not appear in this table. For some the table may be made to serve. For instance, suppose we want to calculate $328 \div 1,288$. We may take $327.68 \div 1,280$ as an approximate value. Now $1,280 = 10 \times 128 = 10 \times 2^7$, and

$$327.68 = \frac{32768}{100} = \frac{2^{15}}{100},$$

so that the quotient is $\frac{2^{15}}{100} \div (10 \times 2^7)$ or $\frac{2^8}{1000}$, which is $\frac{256}{1000}$ or 0.256.

With regard to the number 2^{15} , you know already that the number 15 that shows how many 2's have to be multiplied together is called the index. You know also that the number 2^{15} itself (or 32,768) is called 'the 15th power of 2'. The relation $2^{15} = 32,768$ may also be expressed by saying that 15 is the logarithm of 32,768; the number 2 that forms each of the 15 factors being called the base of the logarithm. The relation is also written $15 = \log_2 32,768$, or, when it is not necessary to show the base of the logarithm, $15 = \log 32,768$.

3. For many numbers our table is useless. A more useful one may be found by using another number instead of 2. Let us take a number not much greater than 1, say 1.01. Calculate the powers of 1.01 up to the index 10, each to 3 decimal places.

Index	Power	Index	Power
1 ...	1.010	6 ...	1.061
2 ...	1.020	7 ...	1.072
3 ...	1.030	8 ...	1.083
4 ...	1.041	9 ...	1.094
5 ...	1.051	10 ...	1.105

The series of powers increases so gradually that any number between 1.010 and 1.105 may, with tolerable

accuracy, be represented by the nearest number on the table. By extending the table to higher indices we can represent other numbers. How far must we extend the table?—Up to the index that gives 10 as the power, for by shifting the decimal point any number may be made to lie between 1 and 10. Thus 827 would be treated as 3.27×100 , and 0.827 as $3.27/10$.

4. Is the index 100 far enough to go? Find out by calculating the powers for indices of 20, 30, 40, 50, 100. How will you calculate 1.01^{20} ? Need you calculate the power for every index up to 20?— 1.01^{20} is the product of 20 factors, which we can divide into two sets of 10 factors. We know the product of 10 factors to be 1.105, so that the product of 20 factors is 1.105×1.105 . For the rest

$$\begin{aligned} 1.01^{30} &= 1.01^{20} \times 1.01^{10}, & 1.01^{40} &= 1.01^{30} \times 1.01^{10}, \\ 1.01^{50} &= 1.01^{40} \times 1.01^{10}, & 1.01^{100} &= 1.01^{50} \times 1.01^{50}. \end{aligned}$$

If you calculate a succession of numbers, each to 3 decimal places, and deducing each number from the somewhat inaccurate preceding number, can you expect all the numbers to be correct to 3 decimal places? Find, by first calculating the powers for indices 20, 30, 40, 50, 100 in the way proposed, and then calculating each independently by the binomial theorem*.—Using the above relations to calculate the powers, and expressing each as accurately as 3 decimal places allow, we get

Index	10,	20,	30,	40,	50,	100,
Power	1.105,	1.221,	1.849,	1.491,	1.648,	2.716.

The binomial theorem gives

$$\begin{aligned} 1.01^{10} &= 1 + 10 \times 0.01 + \frac{10 \times 9}{2} \times 0.01^2 + \frac{10 \times 9 \times 8}{2 \times 3} \times 0.01^3 + \dots \\ &= 1 + 0.1 + 0.0045 + 0.00012 + \dots \\ &= 1.105 \end{aligned}$$

as nearly as 3 decimal places can express it.

* Return now to chapter X, article 16, and work through to the end of that chapter.

Why do you take only four terms of the series? What are the values of the 5th and 6th terms? The other powers of 1.01 being also calculated by the binomial theorem we have the table

Index	10,	20,	30,	40,	50,	100,
Power	1.105,	1.220,	1.348,	1.489,	1.645,	2.705,

showing that the error of the former method mounts so rapidly that it gives 1.01^{100} correct to only one decimal place.

You must then go beyond the index 100 to get a power greater than 10. Give some idea how much further you must go.—Successive calculation is slow and inaccurate. We use the binomial series and find

$$1.01^{200} = 7.316,$$

$$1.01^{300} = 19.79,$$

showing that we need not go so far as the index 300.

5. A class of 30 boys might easily complete the table, each of them calculating less than 10 entries. One would calculate the entries for indices 11, 41, 71, . . . , stopping at the first power that is beyond 10. The second would take indices, 12, 42, 72, . . . , the third 13, 43, . . . , and so on. How would the first boy calculate his entries, and how may he check them?—He calculates each by the binomial series. As a check he calculates them without the use of the binomial series. Thus 1.01^{11} being $= 1.01^{10} \times 1.01$, he takes the value 1.105 of 1.01^{10} and multiplies by 1.01, finding $1.01^{11} = 1.116$. Further, he knows $1.01^{30} = 1.348$, and that

$$1.01^{11} \times 1.01^{20} = 1.01^{41}, \quad 1.01^{41} \times 1.01^{30} = 1.01^{71}, \dots;$$

so he finds the remaining entries by multiplying 1.116 repeatedly by 1.348. The results will be below the true values, but near enough to expose any serious error.

A few of these entries follow for the sake of illustrative calculations.

Index	Power	Index	Power
41 ...	1.504	201 ...	7.389
42 ...	1.519	202 ...	7.463
43 ...	1.534	203 ...	7.538
44 ...	1.549	204 ...	7.613
45 ...	1.565	205 ...	7.689
46 ...	1.580	206 ...	7.766
47 ...	1.596	207 ...	7.844
48 ...	1.612	208 ...	7.922
49 ...	1.628	209 ...	8.001
50 ...	1.645	210 ...	8.081

6. By means of the entries of the table already given calculate as far as you can

$$(1) 7.76 \times 1.58$$

$$(5) 107 \times 793$$

$$(2) 7.72 \div 1.52$$

$$(6) 793 \div 10.7$$

$$(3) 1.616 \times 1.088$$

$$(7) 7.40 \times 7.57$$

$$(4) 1.616 \div 1.088$$

$$(8) 1,525 \div 780$$

(1) 7.76 is approximately 1.01^{206} , and 1.58 is 1.01^{46} , so that 7.76×1.58 is 1.01 multiplied by itself 252 times, or 1.01^{252} . The entry for index 252 is necessary for the evaluation of this.

(2) $7.72 \div 1.52 = 1.01^{205} \div 1.01^{42} = 1.01^{163}$, and we have not the corresponding entry.

(3) $1.616 \times 1.088 = 1.01^{48} \times 1.01^8 = 1.01^{56}$, and we have not the corresponding entry.

(4) $1.616 \div 1.088 = 1.01^{48} \div 1.01^8 = 1.01^{40} = 1.489$, as this time we happen to have the entry.

(5) $107 \times 793 = 10,000 \times 1.07 \times 7.93 = 10,000 \times 1.01^7 \times 1.01^{208} = 10,000 \times 1.01^{215}$, to evaluate which we should look out the power for index 215 and multiply by 10,000.

(6) $793 \div 10.7 = 10 \times 7.93 \div 1.07 = 10 \times 1.01^{208} \div 1.01^7 = 10 \times 1.01^{201} = 78.89$, as we happen to have the entry.

(7) $7.40 \times 7.57 = 1.01^{201} \times 1.01^{203} = 1.01^{404}$; even our complete table would not give this, as it does not contain indices even so high as 300.

(8) $1,525 \div 780 = 10 \times 1.525 \div 7.80 = 10 \times 1.01^{42} \div 1.01^{206} = 10 \div 1.01^{164}$. If we had the entry for index 164 it would not help here. The division would still remain to be performed, and the sole object of our table is the avoidance of multiplication and division.

7. Can you suggest any way out of the difficulty found in using the table to calculate 7.40×7.57 and $1,525 \div 780$? Can you make use of the last entry for which the power is 10?—If the last index on the table is k , so that $1.01^k = 10$, we can write 1.01^{404} in the form $1.01^k \times 1.01^{404-k}$ or $10 \times 1.01^{404-k}$, and the table will contain $404 - k$.

You know that k lies between 200 and 300. To find more exactly where it lies calculate the entries for indices 210, 220, . . .—They are:—

Index	Power
210 ...	8.081
220 ...	8.927
230 ...	9.861
240 ...	10.898

Now calculate the entries for 231, 232, . . .—They are:—

231 ...	9.959
232 ...	10.059

The number 10 lies between these two powers, so we need go no further. And of the two 9.959 is nearer to 10, so we take the approximate result,

$$1.01^{231} = 10.$$

Now calculate 1.01^{404} .—It is $1.01^{231} \times 1.01^{173}$, or 10×1.01^{173} , and the complete table would give this. Thus the table serves for such products as 7.40×7.57 .

Can you now calculate $1,525 \div 780$?—We found this to be $10 \times 1.01^{42} \div 1.01^{206}$, and we want the dividend to have a greater index than the divisor. We replace 10 by 1.01^{231} and have the form $1.01^{273} \div 1.01^{206}$, or 1.01^{67} , for which form the table is useful.

8. The relation between power and index may be shown on a graph. As data from which to draw the graph it is

sufficient to calculate every twentieth entry of the table. (For the data see p. 374.) Draw the graph, using scales as large as possible, for the larger your scales the more accurate will your results be. You can save some space by showing the power-scale from 1 to 10 only, as none of the powers we use fall between 0 and 1.

Use the graph to make the calculations of Art. 6.

(1) We found that $7.76 \times 1.58 = 1.01^{206} \times 1.01^{46} = 1.01^{252}$. This index is greater than 231, so we cannot read the corresponding power from the graph. We put it in the form $1.01^{231} \times 1.01^{21}$ or 10×1.01^{21} . The graph gives the value of 1.01^{21} as 1.23, so that the product sought is 12.3.

(2) $7.72 \div 1.52$. Mark the point P that represents 7.72 on the axis of powers or ordinates (Fig. 1), and draw PQ parallel to the axis of indices or abscissae. The length of PQ is the index of 7.72. In the same way get the index RS of 1.52. Then we know that $7.72 \div 1.52 = 1.01^{PQ} \div 1.01^{RS}$, that is, $1.01^{205} \div 1.01^{42}$; and the quotient of 205 factors divided by 42 factors is 163 factors, or 1.01^{163} . The graph gives the point U or 5.06 as the value of 1.01^{163} .

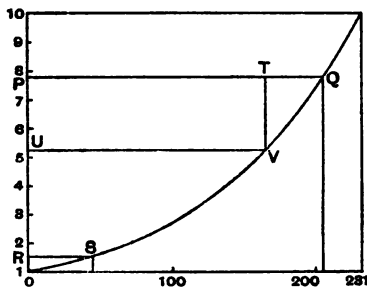


FIG. 1.

We can avoid the actual reading of the index scale. Having drawn PQ and RS , we cut from PQ a part QT equal to RS , and draw TV parallel to the power axis. Then draw VU parallel to the index axis, and we get U or 5.06 as the quotient.

(3) 1.616×1.088 . In this case we find two indices PQ and RS , which must be added. So we lay in the curve an abscissa of the combined length of the two lines. The foot of this abscissa gives the power 1.76 as the product of 1.616 and 1.088.

(4) $1.616 \div 1.088$. This is like No. 2, and the result 1.49.

that our unit for the index scale is 1 millimetre, so that the greatest abscissa is 231 mm. long. Read off from the graph the indices corresponding to powers 1, 2, 3, ... 10.— They are:—

Power 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Index 0, 70, 110, 139, 162, 180, 196, 209, 221, 231.

Make dots at distances 69, 110, ... mm. along the index axis; but instead of marking them with these distances, mark them with the numbers 1, 2, 3, ... 10 (Fig. 3, sketch).

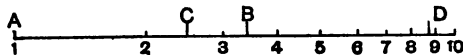


FIG. 3.

Further, make dots on the index axis corresponding to the powers 1.1, 1.2, 1.3, ... 1.9; 2.1, 2.2, ...; but there will be no room to write these powers beside the dots.

10. Can you use this scale (shown reduced in Fig. 3) in place of the graph to calculate 3.4×2.6 ?—Denote by B the dot corresponding to the power 3.4 , then the length AB (in millimetres) is the index corresponding to 3.4 ; or $3.4 = 1.01^{AB}$. Denote by C the dot corresponding to the power 2.6 , so that the length AC is the index of 2.6 , and $2.6 = 1.01^{AC}$. Now mark off BD of the same length as AC , and we have $3.4 \times 2.6 = 1.01^{AB} \times 1.01^{AC}$, or $= 1.01^{AB} \times 1.01^{BD}$, or $= 1.01^{AD}$. And we know that the point D is marked on the scale, not by its distance from A , but by the corresponding power. D is close to the dot that corresponds to the power 8.8, so that $1.01^{AD} = 8.8$ and $3.4 \times 2.6 = 8.8$.

How would you calculate 3.48×2.68 ?—The graph is practically straight from 3.4 to 3.5, so that the divisions of Fig. 3 for 3.41, 3.42, ... 3.49 would be practically equally spaced if there was room to put them in. We judge the position of B for 3.48 by eye, and so for C . Then if D lies between two dots we estimate by eye how it divides the interval, and so get the second decimal place in the product.

Ex. 1. Show that if the graph was actually straight between the points 3.4 and 3.5 the divisions of Fig. 3 for 3.41, 3.42, ... would be actually equally spaced.

11. How will you mark off BD equal to AC ?—It could be done by opening dividers to the length AC , and so transferring it. Would it help to have a second copy of the scale?—We could then place the point A of the second scale against B (Fig. 4), and C on the second scale would then lie against the point D that we want.

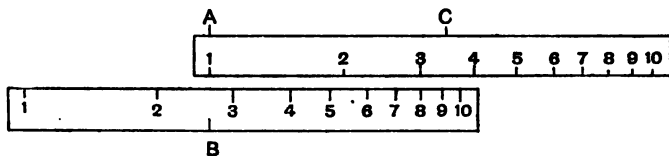


FIG. 4.

Make two such scales on strips of paper, and use them to calculate 8.43×2.68 . Use them also to calculate $8.43 \div 2.68$.

12. Use your two scales to calculate 9.19×2.68 .— B being the point marked 9.19 on the first scale, C the point 2.68 on the second, we put the beginning of the second scale against B (Fig. 5). We know that $1.01^{40} = 9.18 \times$

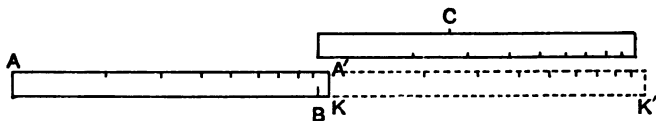


FIG. 5.

2.68 ; but C is beyond the end K of the first scale, that is, AC is greater than 231 . $1.01^{40} = 1.01^{4K} \times 1.01^{K0} = 10 \times 1.01^{K0}$, so we slide the first scale along till A lies where K was, to the position $A'K'$. The point on the first scale now opposite C is marked 2.46 , so that $1.01^{K0} = 2.46$, and $9.19 \times 2.68 = 24.6$.

The sliding of the first scale to its new position is likely to lead to inaccuracy. Can you suggest an improvement?—We could make the first scale on a strip of twice the length, and repeat the graduations on the added length. The added length would then be ready to hand in the position $A'K'$ whenever it was needed.

Your two strips of paper are now (Fig. 6) a simple form of the instrument called a slide-rule. Use your slide-rule to find all the products and quotients of Art. 6.

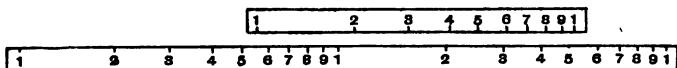


FIG. 6.

13. Such relations as $1\cdot01^{11} \times 1\cdot01^{30} = 1\cdot01^{41}$ have frequently turned up. Show once more the truth of this relation and generalize it.—Since $1\cdot01^{11}$ means $1\cdot01$ multiplied by itself 11 times, the relation written merely says that the product of 11 factors multiplied by the product of 30 factors is the same as if we had taken all 41 factors at once and multiplied them together. What the particular numbers of factors in the two sets are matters nothing, so that we know

$$1\cdot01^{11} \times 1\cdot01^a = 1\cdot01^{11+a},$$

and

$$1\cdot01^b \times 1\cdot01^a = 1\cdot01^{a+b}.$$

Nor does it matter whether the number that forms every factor is $1\cdot01$ or not. Any number k will do, so that we know that

$$k^a \times k^b = k^{a+b}.$$

What kind of numbers may a , b , k be?— a and b are numbers of factors, and must be whole numbers. k may be fractional.

14. We had also such relations as

$$1\cdot01^{10} \times 1\cdot01^{10} = 1\cdot01^{20},$$

$$1\cdot01^{10} \times 1\cdot01^{10} \times 1\cdot01^{10} = 1\cdot01^{30}.$$

Justify these relations, and give a shorter way of writing them.—The first relation says that the product of 10 factors multiplied by the product of 10 factors is the same as the product of the 20 factors taken all together. The second relation says we get the same result by multiplying together 30 factors as by first separating them into batches of 10, finding the product of each batch, and then multiplying the three products together. Just as $1\cdot01 \times 1\cdot01$ is more

shortly written $1\cdot01^2$, the 2 showing how many factors $1\cdot01$ there are; so $1\cdot01^{10} \times 1\cdot01^{10}$ may be written $(1\cdot01^{10})^2$, the 2 showing how often the factor $1\cdot01^{10}$ occurs. In the same way $(1\cdot01^{10})^3$ means the product of 3 factors, each factor being $1\cdot01^{10}$. So that we may write the relations in the form

$$(1\cdot01^{10})^2 = 1\cdot01^{20},$$

$$(1\cdot01^{10})^3 = 1\cdot01^{30}.$$

15. Can you generalize these relations?—We might have any number of factors instead of 2 or 3. If we have a factors, each being $1\cdot01^{10}$, each of these a factors contains the factor $1\cdot01$ 10 times, so that the factor $1\cdot01$ occurs altogether $a \times 10$ times; so that

$$(1\cdot01^{10})^a = 1\cdot01^{a \times 10}.$$

It matters nothing that there are 10 factors in each batch. The reasoning applies equally to any number b , so that we know

$$(1\cdot01^b)^a = 1\cdot01^{a \times b}.$$

And nothing depended on the particular number $1\cdot01$ that forms each factor. Any number k , integral or fractional, does equally well, so that we know

$$(k^b)^a = k^{b \times a}.$$

What sort of numbers must a and b be?—They are whole numbers, as they show the number of factors. We know what $1\cdot01^7$ means and what $1\cdot01^6$ means, but we have no meaning for $1\cdot01^{6\frac{1}{2}}$.

16. Can you generalize $1\cdot01^{30} \div 1\cdot01^{17} = 1\cdot01^{13}$?—As before, we show that

$$1\cdot01^a \div 1\cdot01^{17} = 1\cdot01^{a-17},$$

where of course a must be greater than 17. We show further that

$$1\cdot01^a \div 1\cdot01^b = 1\cdot01^{a-b},$$

and

$$k^a \div k^b = k^{a-b},$$

a being greater than b .

How would you express $1\cdot01^{17} \div 1\cdot01^{30}$ simply?—It is 17 factors multiplied together, and then divided by 30 factors. The result is unaltered if we omit the multiplica-

tion by the 17 factors, and at the same time omit the division by 17 of the 30 factors. There remains simply division by 13 factors, which we express by $1 \div 1 \cdot 01^{13}$.

What becomes of the relation $1 \cdot 01^a \div 1 \cdot 01^{17} = 1 \cdot 01^{a-17}$ when $a = 18$, and when $a = 17$?—When $a = 18$ we have 18 factors divided by 17 factors, so that one is left, and the quotient is $1 \cdot 01$. When $a = 17$ we have 17 factors divided by 17 factors, so that the quotient is 1. If you take the relation $1 \cdot 01^a \div 1 \cdot 01^{17} = 1 \cdot 01^{a-17}$, and forget for a moment that it has any meaning, and put 18 for a , what does it become?—It becomes $1 \cdot 01^{18} \div 1 \cdot 01^{17} = 1 \cdot 01^1$. Hitherto, we have not met the index 1. Can you give it a meaning?—The index shows how many factors there are. The index 1 says there is only 1 factor, so that $1 \cdot 01^1$ means simply $1 \cdot 01$.

17. Take again the relation $1 \cdot 01^a \div 1 \cdot 01^{17} = 1 \cdot 01^{a-17}$, and ignoring its meaning write 17 for a .—The relation becomes $1 \cdot 01^{17} \div 1 \cdot 01^{17} = 1 \cdot 01^0$. What do you make of this, with a zero for an index?—An index zero would seem to say that there are no factors. We know that $1 \cdot 01^{17} \div 1 \cdot 01^{17}$ is 1. Clearly, then, if $1 \cdot 01^0$ is to have any meaning it must mean 1. Is it possible that $1 \cdot 01^0$ might turn up somewhere where its meaning had to be something else?—In expressions such as $1 \cdot 01^a \div 1 \cdot 01^b$, $1 \cdot 01^0$ can only turn up when a and b are equal, and then $1 \cdot 01^a \div 1 \cdot 01^b = 1$, so that in all such cases 1 is the appropriate meaning of $1 \cdot 01^0$.

What meaning would you give to k^0 ?—This symbol will turn up from $k^{17} \div k^{17}$ or $k^a \div k^a$, or such expressions, so its appropriate meaning is 1.

18. Once more write $a = 30$ in the relation $1 \cdot 01^{17} \div 1 \cdot 01^a = 1 \cdot 01^{17-a}$.—It becomes $1 \cdot 01^{17} \div 1 \cdot 01^{30} = 1 \cdot 01^{-13}$. Here is another new kind of index, a negative index. Can you give it a meaning?—We know that multiplying together 17 factors and then dividing by 30 factors amounts simply to dividing 1 by 13 factors. So, while $1 \cdot 01^{13}$ means that 13 factors are to be multiplied together, $1 \cdot 01^{-13}$ must mean that 1 is to be divided by 13 factors. Is it possible that $1 \cdot 01^{-13}$ might turn up where it would have to have some other meaning?—It can only turn up when

there are 13 more factors in the divisor than in the dividend, so that the meaning already given is appropriate.

What meaning would you give to k^{-a} ?

19. We have sometimes wanted to find the square root of a number, that is, to find another number which, multiplied by itself, gives the original number. Hitherto, our only means has

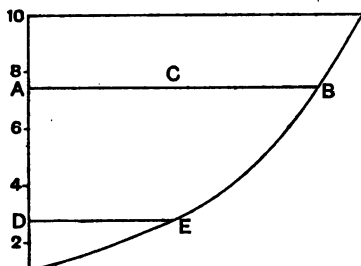


FIG. 7.

AB to the curve (Fig. 7). We bisect AB in C , and lay in the curve the abscissa DE , whose length is AC . The power 2.73 represented by D is the square root. For $2.73 = 1.01^{.28}$, so that $2.73 \times 2.73 = 1.01^{.28} \times 1.01^{.28} = 1.01^{.56} = 7.48$.

Could you use a complete table of powers of 1.01 to find a square root?—The table shows that 7.48 lies between 1.01^{202} and 1.01^{203} . Now $1.01^{101} \times 1.01^{101} = 1.01^{202}$, so that 1.01^{101} is the square root of 1.01^{202} or 7.48 , and we can look its value out in the table. We might have taken 1.01^{203} as the value of 7.48 , but we could not separate the 203 factors into two equal batches. To find a square root we must use only even indices.

20. We had such relations as $1.01^a \times 1.01^a = 1.01^{a \times 2}$, which are true so long as a is a whole number. Let us for a moment forget the meaning of this relation and put $\frac{1}{2}$ instead of a . What does the relation become?—It becomes $1.01^{\frac{1}{2}} \times 1.01^{\frac{1}{2}} = 1.01$. If $1.01^{\frac{1}{2}}$ is to have any meaning it must mean the number which, multiplied by itself, makes 1.01 , that is, the square root of 1.01 .

Calculate the squares of $1.001, 1.002, 1.003, \dots$. Do these give you the square root of 1.01 ?— $1.005 \times 1.005 = 1.010025$, so that 1.005 is pretty accurately the square root of 1.01 .

In the relations $1\cdot01^a \times 1\cdot01^b = 1\cdot01^{a+b}$, $1\cdot01^a \div 1\cdot01^b = 1\cdot01^{a-b}$, $(1\cdot01^a)^b = 1\cdot01^{a \times b}$ give a and b various values such that $\frac{1}{2}$ appears as an index, and see if the relations hold on the supposition that $1\cdot01^{\frac{1}{2}}$ means the square root of $1\cdot01$, namely, $1\cdot005$.—If $a = 1$, $b = \frac{1}{2}$ the relations become $1\cdot01 \times 1\cdot01^{\frac{1}{2}} = 1\cdot01^{\frac{3}{2}}$, $1\cdot01 \div 1\cdot01^{\frac{1}{2}} = 1\cdot01^{\frac{1}{2}}$, $1\cdot01^{\frac{1}{2}} = 1\cdot01^{\frac{1}{2}}$. Whether the first of these is true we cannot say, because we have no meaning for $1\cdot01^{\frac{3}{2}}$; and the second is another way of saying that $1\cdot01^{\frac{1}{2}}$ multiplied by itself is equal to $1\cdot01$. If $a = \frac{1}{2}$, $b = 1$ the first relation is, as before, $1\cdot01^{\frac{1}{2}} \times 1\cdot01 = 1\cdot01^{\frac{3}{2}}$. The second becomes $1\cdot01^{\frac{1}{2}} \div 1\cdot01 = 1\cdot01^{-\frac{1}{2}}$, and this cannot be called either true or untrue until $1\cdot01^{-\frac{1}{2}}$ is given a meaning. The third relation becomes $1\cdot01^{\frac{1}{2}} = 1\cdot01^{\frac{1}{2}}$. So far we have found no case in which the meaning proposed for $1\cdot01^{\frac{1}{2}}$ is unsuitable.

21. It is in some ways better to write $\frac{1}{2}$ in decimal form $0\cdot5$. In the relation $(1\cdot01^a)^b = 1\cdot01^{a \times b}$ give a the value $\frac{1}{2}$ and b various integral values.—Giving b the value 2, we have $(1\cdot01^{0\cdot5})^2 = 1\cdot01$, which is another way of saying that $1\cdot01^{0\cdot5}$ multiplied by itself makes $1\cdot01$. Putting $b = 3$ we have $(1\cdot01^{0\cdot5})^3 = 1\cdot01^{1\cdot5}$, but we have no value for $1\cdot01^{1\cdot5}$. Putting $b = 4$ we have $(1\cdot01^{0\cdot5})^4 = 1\cdot01^2$. To see if this is true we calculate each term. Taking $1\cdot005$, the square root of $1\cdot01$, as the value of $1\cdot01^{0\cdot5}$ we have $(1\cdot01^{0\cdot5})^4 = 1\cdot005^4 = 1\cdot02$; and we know that $1\cdot01^2 = 1\cdot02$; so the relation is true. Or we may reason otherwise; whatever number k is we know that $k^4 = k^2 \times k^2$; and $(1\cdot01^{0\cdot5})^4 = (1\cdot01^{0\cdot5})^2 \times (1\cdot01^{0\cdot5})^2$, and this is equal to $1\cdot01 \times 1\cdot01$; that is, the relation is true.

These tests justify us (as far as mere experiment can) in taking $1\cdot01^{\frac{1}{2}}$ or $1\cdot01^{0\cdot5}$ to mean the square root of $1\cdot01$, and further, in assuming that the relations $1\cdot01^a \times 1\cdot01^b = 1\cdot01^{a+b}$ and $(1\cdot01^a)^b = 1\cdot01^{a \times b}$ hold even when $0\cdot5$ appears among the indices.

Generalize this.—We agree that $k^{0\cdot5}$ is to mean the square root of k , and trial shows, as in the case of $1\cdot01$, that the laws $k^a \times k^b = k^{a+b}$ and $(k^a)^b = k^{a \times b}$ hold even when $0\cdot5$ appears among the indices.

Another way of writing the square root of 10 or of k is $\sqrt{10}$, $\sqrt{k}$.

EXERCISES.

2. *A, B, C, D* are four stations on a railway; the distance *AB* is 10 miles; *BC*, 10; *CD*, 8. The following is an extract from a time-table:—

Up.		Down.	
<i>A</i>	dep. 7.57 a.m.	<i>D</i>	dep. 8.29 a.m.
<i>B</i>	{ arr. 8.15 a.m.	<i>C</i>	—
	{ dep. 8.18 a.m.		
<i>C</i>	{ arr. 8.37 a.m.	<i>B</i>	—
	{ dep. 8.40 a.m.		
<i>D</i>	arr. 8.55 a.m.	<i>A</i>	arr. 9.10 a.m.

Plot in one diagram on squared paper graphs to show the positions of both trains at any time between those given, assuming that they run uniformly between the stations. When, and where, do they pass one another?

3. The water supply of a village of 5,700 inhabitants is brought by a pipe 150 square inches in section, and the water flows along the pipe steadily at 2 miles an hour. Give, to the nearest ten thousand gallons, the total daily water supply, and also, to the nearest ten gallons, the daily supply per head. A gallon is 277 cubic inches.

4. A number of girls are to be distinguished by wearing different rosettes. Ribbons of three colours are available to make the rosettes, each rosette may be of one colour, of two, or of three, and no two rosettes are to be the same. For how many will these three colours serve?

If another colour was added, making four, how many more girls could be included?

5. Calculate the value of $\frac{P}{r} \left(1 - \frac{1}{(1+r)^n}\right)$, where $P = £454$, $r = 0.02$, $n = 24$, the value of P being given to the nearest hundredth, those of r and n exactly.

6. Calculate, to two decimal places, the value when $x = 1$ of

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$$

where $3!$ means $1 \times 2 \times 3$, $5!$ means $1 \times 2 \times 3 \times 4 \times 5$, &c.

7. A cask is 4 feet 6 inches deep, its greatest diameter is 2 feet 3 inches, and the diameter of each end is 2 feet. Calculate the number of gallons which it will hold. The volume of a cask of depth h feet of which the greatest and least diameters are D and d

feet is approximately $0.7854 \times h \times \left(\frac{D+d}{2} \right)^2$ cubic feet, and a cubic foot is 6.23 gallons.

Also calculate the error caused by taking this formula instead of the following more accurate formula:—Volume of cask of depth h feet of which the greatest and least diameters are D and d feet is

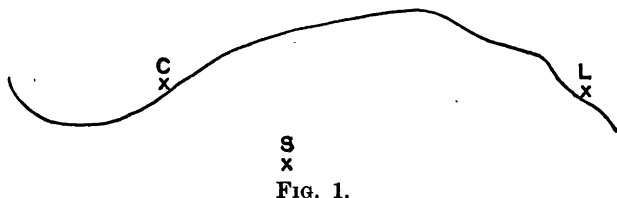
$$0.05236 \times h \times (8 \times D^2 + 4 \times D \times d + 3 \times d^2) \text{ cubic feet.}$$

8. An upright pole 10 feet high casts a shadow 12.6 feet long at midday on a certain day. Another upright pole of the same height, 100 miles further north, casts a shadow 13.2 feet long at the same time. Deduce the earth's perimeter, supposing the earth a sphere.

CHAPTER XIII

DANGER ANGLE. PROPERTIES OF CIRCLES

1. A SHIP'S chart (Fig. 1) shows on a certain coast two points, a church C and a lighthouse L , with the notice that the danger angle is 110° . This means that if a ship S gets so near C and L that the angle CSL is greater than 110° , she is in danger of rocks or shoals. Draw the chart



on double the scale shown in Fig. 1, draw on tracing-paper an angle of 110° , move the tracing-paper about so that the two legs of the angle always pass through C and L , and so draw the curve that separates safe water from unsafe.

Repeat, taking 90° , 70° , 50° in succession as the danger angle. What are the curves?—They appear to be arcs of circles (that is, parts of circles) that pass through C and L .

Supposing one of these curves a circle, how can you find its centre?—We know that the centre lies on the perpendicular bisector of a chord of a circle. By drawing two chords we find the centre as the intersection of the two bisectors.

Find the centres of all these curves. How many positions of S on a danger circle do you need to lay down before you can find the centre of the circle?—One, for by joining it to C and L we get two chords.

Find the centres of the curves in this way.

How many points on a circle fix it? That is, how many

points on the circle do you need to know before you can draw the circle?

2. Take any three points C, S, L (Fig. 2), find the centre O of the circle that passes through them, and draw the circle. Prove that the point O where the $\perp$ bisectors of SC and SL meet is equidistant from C, S , and L . What do you know of the perpendicular bisector of CL ?—It also passes through O since it is the locus of points equidistant from C and L , and O is equidistant from C and L . Generalize the result.—

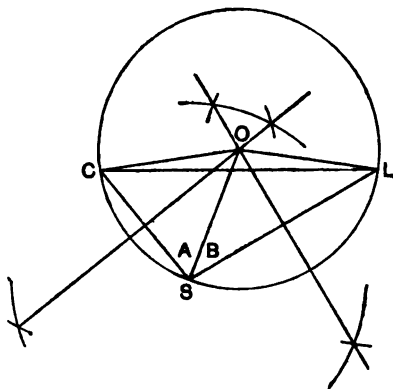


FIG. 2.

The perpendicular bisectors of the three sides of any triangle are concurrent or meet in a point, and the point where they meet is the centre of the circle that passes through the corners of the triangle.

Draw a number of triangles CSL and find how the centre O lies with regard to each. Can you classify the cases?—When the angles of the triangle are all acute O lies inside the triangle; when an angle is obtuse O lies outside and opposite to the obtuse angle.

In each of your figures draw the radii to L, S, C , suppose A the number of degrees in the angle CSO and B the number in $\angle LSO$, and express all the angles of the figure in terms of A and B . How are the angles CSL and COL related?—When $\angle CSL$ is acute $\angle COL = 2A + 2B = 2\angle CSL$; when $\angle CSL$ is obtuse, $\angle COL = 360 - 2A - 2B = 360 - 2\angle CSL$; all the angles being expressed in degrees.

Use these results to find the centre and the radius of a circle of which three points C, S, L , are given.—When $\angle CSL$ is acute we make the two angles CLO and LCO each equal to $90 - \angle CSL$, making O

218 DANGER ANGLE. PROPERTIES OF CIRCLES

lie on the same side as S of CL . When $\angle CSL$ is obtuse, we make the angles CLO and LCO equal to $\angle CSL - 90$, making O lie on the other side of CL from S . In each case O is the centre and OC or OL the radius. Where is O when $\angle CSL$ is right?

3. Suppose the positions of C and L kept fixed while S is varied. Can S move at all without changing the centre O (and in consequence the circle)?—The position of O is settled by the size of the angle CSL , as the construction just given shows; so that S may move without changing O so long as it does not change the size of the angle CSL . That is, if we return to our original idea of a ship's chart, and suppose CSL the danger angle, the position of O is unchanged so long as the church and lighthouse subtend the danger angle at the ship. That is, all positions where C and L subtend the danger angle lie on the same circle.

Are there any points inland at which C and L subtend the danger angle? Do C and L subtend the danger angle at all points on the circle we have found?—If the figure is

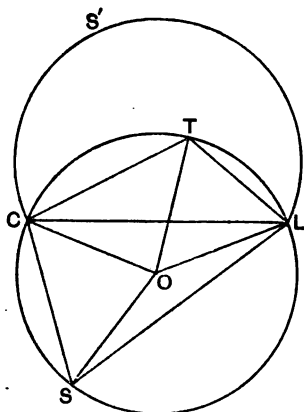


FIG. 3.

folded about CL , every point S on the danger circle falls on another point S' where the same angle is subtended (Fig. 3); in other words, the image in CL of every point S that we have found gives another point S' where CL subtends the danger angle. Again, the angle subtended at any point on the arc CTL is clearly not the danger angle. The complete locus of points where the danger angle is subtended is the two arcs CSL and $CS'L$.

In Fig. 3, draw the radii to C, S, L, T , join TC and TL , and by expressing as many angles of the figure as possible in terms of $\angle CTO$ and $\angle LTO$, find the relation between $\angle CTL$ and $\angle COL$ and between $\angle CTL$ and $\angle CSL$.— $\angle CTL + \angle CSL = 180^\circ$.

4. Draw any circle with centre O (Fig. 4). Suppose a line K to move up across the circle from X to Y . Let K in any position meet the circle in P and Q , and let R be any point above K on the circle. By measuring a number of positions find how the angle PRQ varies as the line K

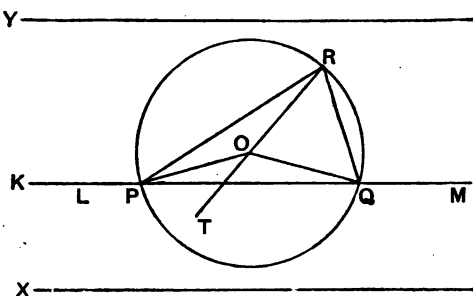


FIG. 4.

moves across the circle.—The angle increases from 0° to 180° . And how does $\angle POQ$ vary?—It increases from 0° to 180° , and then diminishes again to 0° .

What is the relation between $\angle PRQ$ and $\angle POQ$ in any position?—Until the line K passes the centre O , $\angle POQ$ is twice $\angle PRQ$; after that $360^\circ - \angle POQ$ is twice $\angle PRQ$.

Hitherto we have been concerned with angles less than 180° , and in this article we have assumed $\angle POQ$ to mean the angle less than 180° that is contained by OP and OQ . Until the line K reaches the centre O , the 'angle POQ ' means the number of degrees a line must turn through to get from OP to OQ , turning counterclockwise, that is, opposite to the direction in which the hands of a clock turn. Let us continue this meaning for the angle POQ after the centre O is passed, and admit angles of more than 180° . With this meaning how does the angle POQ vary as the line K moves across the circle from X to Y ?— $\angle POQ$ increases from 0° at first contact of the line with the circle to 360° at last contact.

5. In Fig. 4 join RO and produce it to any point T . Draw the figure in all possible forms, $\angle POQ$ less than 180° , or greater than 180° , the $\angle PRQ$ enclosing O or not

enclosing O , or having O on one leg. What is in each case the relation between $\angle TOP$ and $\angle TRP$, between $\angle TOQ$ and $\angle TRQ$?—The angle-sum property of a triangle shows $\angle OPR + \angle ORP + \angle ROP$ equal to 180° , and so equal to $\angle ROP + \angle POT$. So that $\angle POT$ is equal to $\angle ORP + \angle OPR$, or to twice either of them, since they are equal. In the same way $\angle TOQ = 2\angle TRQ$. And how is $\angle POQ$ related to $\angle PRQ$?—In every case by addition or subtraction of $\angle TOP$ and $\angle TOQ$, and of $\angle TRP$ and $\angle TRQ$, we find $\angle POQ = 2\angle PRQ$; $\angle POQ$ meaning the angle through which a line turns counterclockwise to get from OP to OQ .

6. Any piece of the circumference of a circle is called an arc. The area enclosed between an arc and the chord joining its ends is called a segment. Thus in Fig. 5,

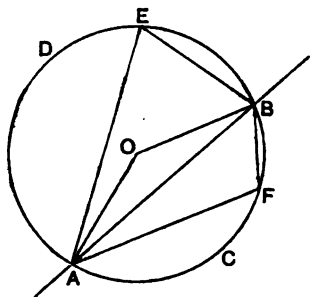


FIG. 5.

ADB is an arc, and the space bounded by the arc ADB and the chord AB is a segment. Further, the angle AEB is said to be an angle in the segment ADB , or an angle at the circumference standing on the arc ACB . The two angles between OA and OB , one less than 180° and one greater than 180° , may be distinguished as the angle at O standing on the arc ACB and the angle at O standing on the

arc ADB . By means of these terms state the conclusion arrived at.—The angle at the centre of a circle is double the angle at the circumference that stands on the same arc, and the angle at the circumference on a given arc is the same for all points along the circumference. Or, in other words, all the angles in the same segment of a circle are equal, each being half of the angle at the centre that stands on the same arc. Thus in Fig. 5 the acute angle AEB and the angle AOB less than 180° stand on the same arc ACB , while the obtuse angle AFB and the angle AOB greater than 180° stand on the same arc ADB .

Ex. 1. Justify the following method of drawing a $\perp$ at a point A of a line X :—With any centre B and radius BA draw a circle cutting the line X again in C . Draw the diameter CBD that passes through C . Then AD is $\perp$ to the line X .

7. In Fig. 4, measure the angles POQ , OPL , OQM . When the angle POQ contains a degrees, what are the values of the other two angles?— $\angle$ s OPL and OQM are each $90 + a/2$.

Suppose the line K to move downwards from the position shown to X . How does the number of its intersections with the circle vary?—The number is two for a while, then one and then none.

Give the angles OPL and OQM when $\angle POQ$ is 1 degree, and when 0.01 degree.—Each of the angles is 90.5 degrees in the first case, and 90.005 in the second. As the line K moves downwards, the radii OP and OQ become more and more nearly perpendicular to K as the angle POQ becomes smaller and smaller. We can suppose POQ as small as we please, and so the radii OP and OQ as nearly $\perp$ to K as we please.

8. Two lines RO and RE meet in R (Fig. 6). How will you find whether a circle with centre O and radius OR will cut RE in any other point than R ?—By drawing the circle.

Is there any case in which much depends on the accuracy of your drawing? Can you by general reasoning discriminate the cases in which the circle cuts RE again?—Draw (or suppose drawn) $ON \perp$ to RE . Mark off NS equal to NR .

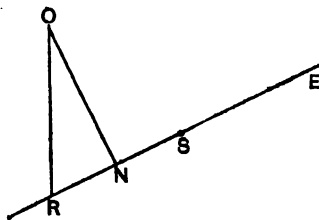


FIG. 6.

The triangles ONR and ONS are congruent, and $OS = OR$; so that the circle must cut RE again at S .

Does the circle then in all circumstances cut RE again?—Always, except when R is itself the foot of the $\perp$ from O on RE .

Can you prove that when R is the foot of the $\perp$ the circle does not cut RE again?—We know that any other point E is further from O than the foot of the $\perp$ (chap. XI, art. 3), so that any such point E is further from the centre than any point on the circle.

Such a line that is $\perp$ to a radius of a circle and meets the circle in one point only, or 'touches' the circle, is (inconveniently enough) given a name that you have met already in a different sense. It is called a tangent to the circle.

9. It was found that in Fig. 3 $\angle$ s CSL and CTL made up together 180° . State the result as generally as you can, and prove it.—The sum of a pair of opposite angles of a quadrilateral inscribed in a circle is 180° . For each angle is half that at the centre on the same arc, and the two angles at the centre make up 360° .

State and prove a converse proposition.—If a quadrilateral has the sum of two opposite angles equal to 180° , a circle can be drawn about it, that is, passing through its four corners. Suppose that in Fig. 7 $\angle$ s S and Q of $PQRS$ are together $= 180^\circ$. Find the centre O of the circle through PQR as the intersection of the $\perp$ bisectors of PQ and QR , and draw the circle. The locus of points where PR subtends an angle $180 - Q$ is the arc PTR of this circle and the image in PR of the arc. So S lies on one of these arcs. If S lay on the image PUR , say at U , we should have a quadrilateral with the angle PQR greater than 180° . Let us exclude such angles, and S will lie on the arc PTR , and the proposition will be true.

Ex. 2. Draw any quadrilateral in a circle, measure its angles, and so find the sum of each pair of opposite angles.

Ex. 3. Draw a quadrilateral having a pair of opposite angles together equal to 180° . Measure the other two angles and so find their sum. Draw a circle through three of the corners of the quadrilateral and see if it goes through the fourth corner.

10. If the sides of the quadrilateral are produced, thus making exterior angles of the figure, how may these two converse propositions be stated?—A quadrilateral in a circle has the angle between a pair of sides equal to the exterior angle between the other pair of sides; and if a quadrilateral has the angle between one pair of sides equal to the exterior angle between the other pair, a circle can be drawn about it.

In a circle draw a quadrilateral $PQRS$ (Fig. 7), and produce one side RS both ways. Join O , the centre of the circle, to PQR . Suppose $\angle OSP$ to contain a degrees,

$\angle ORQ$ b degrees, and $\angle SOR$ c degrees. Are all the other angles determined when a b c are known?—Yes, for these angles are enough to enable us to draw the whole figure except as to size. Draw a number of figures of different forms and in each figure express all the other

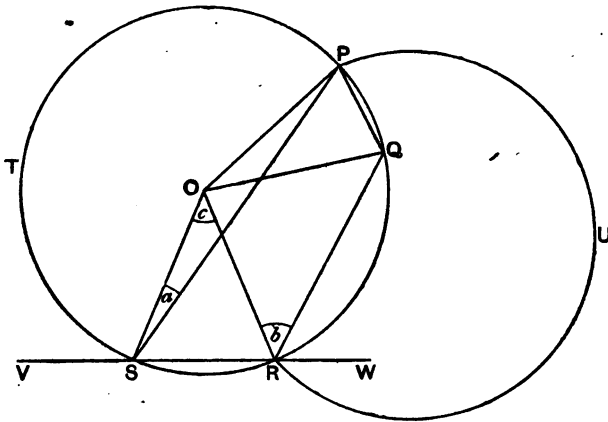


FIG. 7.

angles in terms of a b c . In particular, what values have SPQ , RQP , PSV , QRW ?—In the figure drawn (Fig. 7)

$$OSR = 90 - \frac{c}{2}, \quad \text{so that } PSV = 90 + \frac{c}{2} + a.$$

$$QRW = 180 - ORQ - ORS = 90 + \frac{c}{2} - b.$$

$$\begin{aligned}
 POQ &= POS - QOR - ROS = (180 - 2a) - (180 - 2b) - c \\
 &= 2b - 2a - c; \quad \text{whence } OPQ.
 \end{aligned}$$

$$SPQ = OPQ - OPS = 90 + \frac{c}{2} - b.$$

$$RQP = RQO + OQP = 90 + \frac{c}{2} + a.$$

224 DANGER ANGLE. PROPERTIES OF CIRCLES

Thus the particular results wanted are

$$QRW = SPQ = 90 + \frac{c}{2} - b,$$

and
$$PSV = PQR = 90 + \frac{c}{2} + a.$$

Ex. 4. Check these expressions by measuring a , b , c on one of your figures, substituting their values in the expressions, and comparing the results with their values measured on the figure.

11. Suppose now in each of your figures the angle c to become so small that we may neglect it. Draw the figures again for this case. What do the expressions for the angles become, and what is the form of the figure?—The expressions become $QRW = SPQ = 90 - b$, and $PSV = PQR = 90 + a$ for the figure drawn. S and R come so close together that OS and OR are indistinguishable from $\perp$ s to VW , and VW is indistinguishable from a tangent to the circle.

12. Consider the case in which VW is actually a tan-

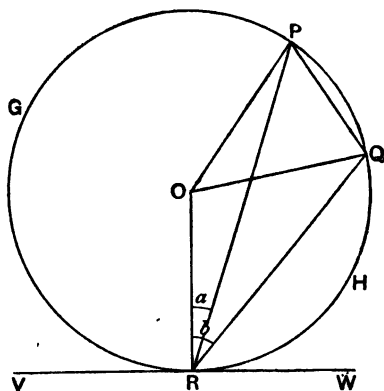


FIG. 8.

gent. Draw any triangle PQR in a circle (Fig. 8), and at R draw $VRW \perp OR$ and therefore a tangent to the circle. Join O to P , Q , R . Call $\angle ORP$ a and $\angle ORQ$ b . Draw figures of all possible forms, and for each figure express all the other angles of the figure in terms of a and b . In particular, what are the angles RPQ , RQP , PRV , QRW ?—From among the expressions for the angles we pick out

$PRV = PQR = 90 + a$, and $QRW = QPR = 90 - b$, for the particular figure drawn (Fig. 8). This agrees with the result found from the quadrilateral by making c very small. State

the result formally.—We know that the angle QPR is equal to any other angle in the segment QGR , and that $\angle PQR$ is equal to any other angle in the segment PHR . So the result may be stated: Each angle between a chord of a circle and the tangent at one end of the chord is equal to the angle in the alternate segment of the circle.

Ex. 5. Verify the result first stated by measuring $\angle s$ RPQ , RQP , VRP , WRQ on each of your figures.

Ex. 6. Measure the angle QVR and the angle in the segment QHR . How would you prove them equal?

Measure the angle PBW and the angle in the segment PGR . How would you prove them equal?

13. Draw a large circle, radius about 10 cm. Lay in it a chord CL , and let a point S travel along one of the arcs into which the circle is divided. Join SL and produce it to Q . Also join L to the centre O of the circle. Find by drawing how the angles CLQ and OLQ vary as S travels along the arc and comes near L . Measure also the angle in the segment CSL .

Again, draw a line CL . On tracing-paper draw two lines, ASB and PSQ , crossing at S . Lay the tracing-paper on CL so that SA passes through C and SP through L . Measure the angle LCA . Move the tracing-paper about (keeping SA through C and SP through L), so that S comes nearer and nearer to C , and see how the angle LCA varies.

Does the preceding discussion help towards finding the values of the various angles as S approaches C or L ?—In the first case of the present article $\angle OLQ$ becomes more and more nearly right, and SLQ becomes more and more nearly a tangent. We know that each angle between CL and the tangent at L is equal to the angle in the alternate segment; so that $\angle CLQ$ becomes more and more nearly equal to the angle in the segment CSL . In the second case we know that S describes an arc of a circle passing through C and L , so that it is essentially the same case as the first.

14. Draw any line CL , and any angle A . Show how, without the use of a protractor, to make on CL a segment of a circle that will contain an angle A .—Make at L an

226 DANGER ANGLE. PROPERTIES OF CIRCLES

angle CLP equal to A (Fig. 9). Draw the $\perp$ bisector of CL , and the $\perp$ at L to LP . The intersection of these $\perp$ s is the centre of the circle.

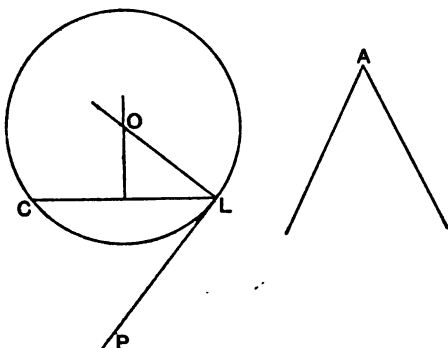


FIG. 9.

Ex. 7. Show how to draw a circle to touch a given line at a given point and to pass through another given point. Is there any case in which it cannot be done?

Rectangle properties of circles.

15. Draw a large circle, centre C , and take a point O lying either inside or outside. Let a point P travel along the circumference and see how the length OP varies. Begin with P where you please; then let it travel round the circle and back to the initial point, so that the radius CP turns through 360° . Give a graph showing the length OP as a function of the angle through which CP has turned. When is OP greatest and when least?

Ex. 8. How does the angle COP vary? What difference according as O is inside or outside the circle?

16. The line OP , considered as unlimited in length, usually cuts the circle in a second point Q . For each position of P measure OQ , and calculate $OP \times OQ$. Give a graph showing $OP \times OQ$ as a function of the angle CP has turned through. What is your experimental result?—The product $OP \times OQ$ is constant.

Ex. 9. For each of a number of positions of CP , draw on squared paper a rectangle with sides equal to OP and OQ , and find the area by counting squares.

17. Draw the diameter AB that passes through O . What relation have you between lengths made on PQ and on this diameter by O and the circle?— $OP \times OQ = OA \times OB$. State this relation as a proportion, that is, as an equality of ratios.—It may be stated

$$\frac{OP}{OA} = \frac{OB}{OQ}, \text{ or } \frac{OP}{OB} = \frac{OA}{OQ}, \text{ or } \frac{OQ}{OA} = \frac{OB}{OP}, \text{ \&c.}$$

What method have you had for proving the equality of ratios? Can you apply it here? Can you find any similar triangles?—Join AQ and BP . In Fig. 10, where O lies inside the circle, the vertically opposite angles AOQ and BOP are equal, the angles QAB and QPB in the same segment $QAPB$ are equal, and the angles AQP and ABP in the same segment $AQBP$ are equal. The triangles AOQ

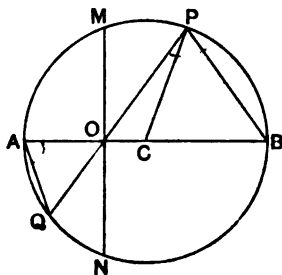


FIG. 10.

and POB are similar, and $\frac{AO}{QO} = \frac{PO}{BO}$. How do you know that these ratios are equal, and not $\frac{AO}{QO}$ and $\frac{BO}{PO}$?

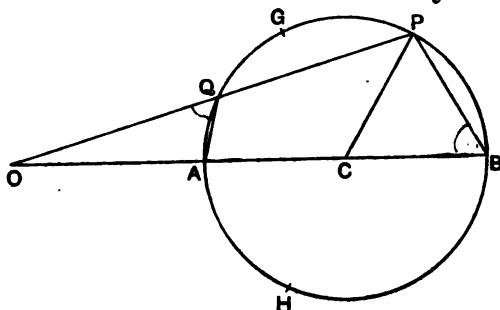


FIG. 11.

Proceed with the case of O outside the circle.—In this

228 DANGER ANGLE. PROPERTIES OF CIRCLES

case (Fig. 11) the angle ABP between two sides of the quadrilateral $ABPQ$ in a circle is equal to the external angle AQO between the other two sides, and in the same way $\angle BPQ = \angle QAO$. So that the Δ s OBP and OQA are similar, and the ratios $\frac{OA}{OQ}$ and $\frac{OP}{OB}$ are equal. How

do you know it is not $\frac{OB}{OP}$ that is equal to $\frac{OA}{OQ}$?

Ex. 10. From measurements of each of your figures calculate $\frac{OA}{OQ}$ and $\frac{OP}{OB}$, and so test the proposition.

Ex. 11. Prove the proposition for two positions of OP , neither position passing through C .

18. The length of the radius being r cm. and the distance of O from the centre being s cm., express OA , OB , and $OA \times OB$ in terms of r and s .—Lengths being in cm. and areas in sq. cm., we have in Fig. 10,

$$AO = CA - CO = r - s, \quad BO = r + s, \\ OA \times OB = (r - s) \times (r + s) = rr - ss, \text{ or } r^2 - s^2.$$

In Fig. 11

$$OA = s - r, \quad OB = s + r, \quad \text{and} \quad OA \times OB = (s - r)(s + r) = s^2 - r^2.$$

Express $OP \times OQ$ in terms of s and r .

Ex. 12. Measure r and s on each of your figures, calculate the difference between r^2 and s^2 , and compare it with the values you found for $OP \times OQ$ for various positions of CP .

Ex. 13. Draw a rectangle with sides $r + s$ centimetres and $r - s$ centimetres, and show how to cut it up and rearrange it as a square r centimetres in the side, with a little square s centimetres in the side cut away.

19. Consider the case of O inside the circle. On the diagram that holds your graph of OP as a function of the angle turned through by CP , draw a graph giving OQ as a function of the angle turned through by CP . For what positions is OQ greater than OP , and for what positions is it less?—Draw the chord $MON \perp$ to AOB . When P lies on the greater arc MBN , OQ is less than OP ; when P lies on the less arc MAN , OQ is greater than OP ; and when P

is at M or at N , OQ is $= OP$.

Express $OM \times ON$ in terms of r and s , and when $r = 9$ and $s = 7$, find the length of OM .— $OM \times ON = r^2 - s^2$. We could find the length of OM by drawing the figure and measuring. Or, CO being the $\perp$ from the centre of the circle on the chord MN we know that OM is $= ON$. So $OM \times OM$, or OM^2 , is equal to $r^2 - s^2$, that is, $81 - 49$, or 32 . From the graph of x^2 (chap. III, art. 17), we find that approximately $5.6^2 = 32$, so that $OM = 5.6$; all distances being in centimetres.

Suppose in general OM to contain k centimetres. Write down the relation between k , r , s .—It is $k^2 = r^2 - s^2$, or $r^2 = k^2 + s^2$, or $s^2 = r^2 - k^2$. If a triangle is made with these lengths k , r , s as sides, what do you know of its shape? Draw it for the case $r = 9$, $s = 7$, $k = 5.6$.—The triangle is right angled, for it is the triangle COM of Fig. 10.

Ex. 14. Prove that if $AOCB$ is a diameter of a circle and OM drawn $\perp$ to $AOCB$ meets the circumference in M , then $OM \times OM = OA \times OB$. (Do not simply quote the main proposition but prove this case by considering similar triangles.)

Ex. 15. Show how to draw a square equal in area to a given rectangle.

Ex. 16. Show how to draw a square equal in area to a given triangle.

Ex. 17. Draw a line representing 10 on any convenient scale. Show how to draw another line to represent on the same scale a number that multiplied by itself is equal to 10.

20. Consider the case of O outside the circle. On the same diagram that contains your graph of OP , give a graph showing OQ as a function of the angle CP has turned through. Distinguish when OP is greater than, less than, and equal to OQ .— Q and P travel round the circle in opposite directions. There are two points G and H at which they pass one another, and these two points divide the circle into two arcs. When P is on the greater arc, $OP > OQ$; when P is on the less arc, $OP < OQ$; and when P is at G or H , $OP = OQ$. What is a line such as OQ or OH that meets a circle in only one point? You have expressed $OP \times OQ$ in terms of r and s . What does

280 DANGER ANGLE. PROPERTIES OF CIRCLES

this relation become at the points G and H where P and Q pass one another?—It is $OG \times OG = s^2 - r^2$.

21. How would you draw a tangent to a circle, centre C , from a point O outside it (Fig. 12)?—We could lay a ruler against the circle and through O and so draw the tangent. That method gives the tangent very well, but does not give the point of contact very accurately.—

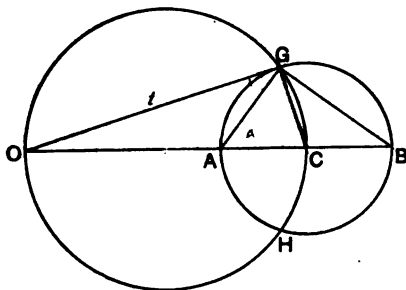


FIG. 12.

G being the point of contact we know that $\angle OGC$ is a right angle, so that G must lie on a circle on OC as diameter. We draw this circle and so find the two points of contact of tangents from O .

By considering similar triangles, find a relation between OG , OA , and OB .—Join GA and GB . The angle OGA between the tangent GO and the chord GA is equal to the angle GBA in the alternate segment. So the two triangles OAG and OBG have $\angle OGA = \angle OBG$, and the angle at O is an angle of each triangle. The triangles are therefore similar, and $\frac{OA}{OG} = \frac{OG}{OB}$, or $OG^2 = OA \times OB$.

22. We saw earlier that the sign of multiplication $\times$ could sometimes be omitted without loss of clearness. In the present case confusion would result from its omission, but a dot may be used without loss of clearness. The above relation is then written $OG^2 = OA \cdot OB$.

An equality of two ratios, e. g. $\frac{OA}{OG} = \frac{OG}{OB}$ in Figs. 10 and

11, is called a proportion, and the four lengths OA , OQ , OP , OB are said to be in proportion. When the proportion contains the same length twice, as a numerator in one ratio and as a denominator in the other, this length is called a mean proportional between the other two. Thus OG in Fig. 12 is a mean proportional between OA and OB .

23. Suppose the length of the tangent OG to be t cm. and give a relation between t , r , s .— $OA = s - r$, $OB = s + r$, $t^2 = OA \cdot OB = s^2 - r^2$. It may also be put in the forms $s^2 = r^2 + t^2$, and $r^2 = s^2 - t^2$. What sort of triangle do the three lengths r , s , t form?—They form the right-angled triangle OGC .

24. Sum up your conclusions with regard to the rectangle properties of a circle.—If a line is drawn through a point O and cuts a circle in P and Q , the area of the rectangle contained by OP and OQ is independent of the direction in which the line is drawn through O . If the circle has a radius of r cm. and O is s cm. from the centre; then the area of the rectangle is $r^2 - s^2$ sq. cm. when O is inside the circle, and $s^2 - r^2$ when O is outside. Also when O is outside, the area of the rectangle is equal to the square on the tangent from O to the circle.

EXERCISES.

18. Draw a circle with a triangle ABC inscribed in it. Draw the creases that appear when (i) B is folded on to C ; (ii) AB is folded on to AC . Do the creases meet on the circle? Test by a geometrical proof.

19. A regular hexagon is inscribed in a circle. Express, in degrees, the value of the angle at the centre subtended by each side.

A paper circle is creased so as to show a diameter AB , and the circle is then folded so that the points A and B are brought together to the centre, two other creases being thus obtained at right angles to AB . Prove that the six points now marked on the circum-

232 DANGER ANGLE. PROPERTIES OF CIRCLES

ference by the three creases are angular points of a regular hexagon. You may show this, if you choose, by construction and measurement, but a general proof should also be given.

20. Three straight pieces of railway track meet at three junctions A, B, C , thus forming a triangle. Where would you place a gun so as to cover the three tracks? If your gun won't carry far enough to cover the three junctions at once, where will you place it to command some part of each track?

21. A gasometer is in the form of a cylinder with a rounded top, and the following dimensions are given: diameter = 28 feet, height at edges 14 feet, height in the middle 16 feet. Draw a section of the gasometer on the scale of 1 inch to 10 feet, and calculate the number of cubic feet of gas that the gasometer will hold. Take the volume of the portion at the top (above 14 feet) as half the volume of a cylinder of the same base and height, and the area of a circle as $3\cdot14$ times the square on its radius.

22. There are shoals off a coast on which stand two prominent objects P and Q . A chart of the coast shows two lines from P and Q meeting at an angle α and marked 'danger angle'. What single observation might be taken on a ship when within possible range of danger to test if it was outside the danger zone?

23. Take three points L, M, N , and suppose them to represent the positions of three lighthouses along a stretch of coast LMN . A ship O is off the coast, and the angles LOM and MON are found to be 80° and 75° . Construct the position of the ship, give no proof, but point out why there is only one solution.

Prove the truth of a property of the circle that you employ.

24. A barometer tube, whose internal diameter is 2 centimetres, is 120 centimetres long and is filled with mercury. Find the weight of the mercury, being given that a cubic centimetre of mercury weighs 13·6 grams, and that the area of a circle is $0\cdot785 \times (\text{diameter})^2$. Also find the cost of the mercury at 6s. 3d. a kilogram.

25. State how you would proceed if you were required to draw about a given circle a quadrilateral, which is to be such that a circle may be drawn about it. Briefly justify your method.

To what extent is the shape of the quadrilateral at your disposal?

26. A canal-lock is to be made 100 feet long and 17 feet wide, and to let a boat pass through up-stream the water in the lock must rise $8\frac{1}{2}$ feet. The sluice to admit the water is to be designed

so as when open to let the water rise the $8\frac{1}{2}$ feet in two minutes. How many gallons per second must the open sluice let through? A cubic foot is 6·23 gallons.

27. A bed of coal 14 feet thick is inclined at 23° to the surface. Calculate the number of tons of coal that lie under an acre (4,840 square yards) of surface. A ton of coal occupies 28 cubic feet. (The 14 feet is to be regarded as a measurement at right angles to the surface of the coal bed.)

28. An arch in the form of an arc of a circle 40 feet in diameter crosses a stream 30 feet wide. Calculate, to the nearest inch, how much the arch rises in the centre, and check your calculation by a careful drawing in which 1 inch represents 10 feet.

29. Draw a circle with centre O and radius 8 cm. Make at O three angles of 50° . Take 5 mm. on your dividers and measure the arcs these angles stand on by stepping the dividers along them. Calculate for each the ratio $\frac{\text{arc}}{\text{radius}}$.

Step round the circumference and calculate $\frac{\text{circumference}}{\text{radius}}$. How many degrees must an angle contain to make the arc it stands on equal to the radius?

30. M is a fixed point in a fixed line KL . A circle of radius 1 inch rolls along KL , and an unlimited straight line MA , passing through M , always touches it. Show the locus of the point of contact. Take M about the middle of the page, and use tracing-paper, drawing the circle and the tangent on it, the circle to be about the middle of the tracing-paper.

31. The height of the Peak of Teneriffe is 12,850 feet; the depression of the horizon from its summit is approximately $2''$. What is the diameter of the earth?

32. Two wheels, 3 feet 6 inches and 6 inches in diameter, are fixed in the same plane with their centres 3 feet apart. Find, in any way you please, the length of belt required to go round them.

33. Draw to a scale of 1 inch to 50 yards the roads in front of Buckingham Palace from the dimensions given in the sketch (Fig. 18). The lengths are in yards. The middle line of the Mall goes through the centre of the memorial enclosure and bisects the front of the palace enclosure at right angles. The middle line of

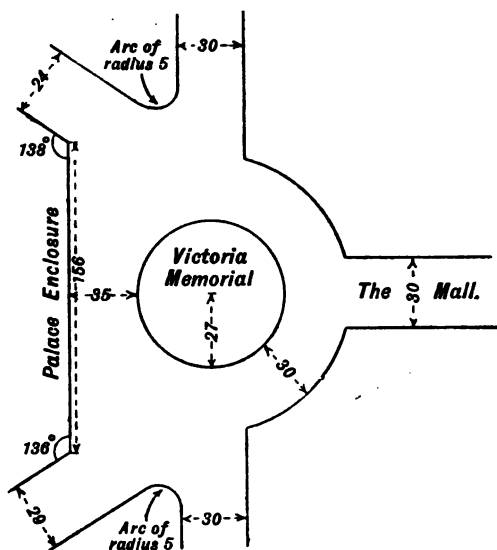


FIG. 13.

two other roads goes through the centre of the memorial enclosure, and is parallel to the front of the palace.

34. Draw a circle of radius 3 inches, and take any point P at a distance of 5 inches from the centre of the circle. Mark four points on the circle at distances 5, 6, 7, 8 inches respectively from P , and join them to P by straight lines. Then make a table like the following and fill it in.

Length of line from P .	Length of line outside circle.	Area of rectangle contained by line and the part outside circle.
5 inches		
6 inches		
7 inches		
8 inches		

What conclusion can you draw from your results? What is the length of the tangent drawn from P to the circle? Draw the tangent.

Prove, for the case in which the cutting line passes through the centre of the circle, that, if from a point outside a circle two straight lines are drawn, one of which cuts the circle and the other touches it, the rectangle contained by the whole line which cuts the circle and the part outside the circle, is equal to the square on the line which touches it.

35. Two straight lines AC and BD cut one another at K so that $\angle AKB = 53^\circ$, $AK = 1.2''$, $BK = 0.7''$, $CK = 1.6''$ and $DK = 2.4''$. A, B, C, D are the angular points of a quadrilateral. Measure the angles of the quadrilateral. Show from your measurements that it is impossible to describe a circle about the figure, and prove the theorem by which you justify your conclusion.

How may the same conclusion be arrived at, from a consideration of the given dimensions of AK, BK, CK and DK ?

36. From a point C outside a circle two straight lines CAB and CT are drawn, CAB passing through the centre and cutting the circle in A and B , CT touching the circle at T . Prove that the square on CT is equal to the rectangle $CA.CB$, and express this relation algebraically, denoting CA by h , the radius of the circle by r , and the length of the tangent by t .

Imagine your figure to represent a section through the centre of the earth, supposed to be a sphere of 4,000 miles radius, and C the top of a cliff. Show that, if the height of the cliff is h feet, the distance of the horizon from C is approximately $\sqrt{\frac{3}{2}h}$ miles.

CHAPTER XIV

LOGARITHMS

1. In discussing the Slide Rule and other methods of speedy calculation (chap. XII) we considered two quantities x and y so related that $y = 1.01^x$. How far would that discussion be altered if instead of 1.01 we used a different number, if for instance we used 1.001?—In this case also we should have a series of indices and corresponding powers. For the powers running from 1 to 10 the series of indices would be different, there would be more entries in our table. The graph made with 1.001 as base would serve the same end in just the same way as the graph made from 1.01. The curve would be the same kind of curve, and would differ only in the graduation of the index axis.

2. Calculate 1.001^{1000} , 1.001^{2000} , 1.001^{3000} , ... far enough to give a rough estimate of the number of entries in our table.—The binomial series for 1.001^{1000} is

$$1 + 1000 \times 0.001 + \frac{1000 \times 999}{2} \times 0.001^2 + \dots$$

$$= 1 + 1 + \frac{0.999}{2} + \frac{0.999}{2} \times \frac{0.998}{8} + \dots,$$

in which we get the fourth term from the third by multiplying by $\frac{0.998}{8}$, the fifth from the fourth by multiplying by $\frac{0.997}{4}$, and so on, the further successive multipliers being $\frac{0.996}{5}$, $\frac{0.995}{6}$, $\frac{0.994}{7}$, To multiply speedily by 0.998, 0.997, ..., we take them in the forms $1 - 0.002$, $1 - 0.003$, The series is $1 + 1 + 0.499 + 0.166 + 0.041 + 0.008 + 0.001$, each term calculated to three places; the remaining terms

do not affect the third decimal place. So the value of $1\cdot001^{1000}$ is 2.715. By calculating $2\cdot715^2$ and $2\cdot715^3$ we find $1\cdot001^{2000} = 7\cdot87$ and $1\cdot001^{3000} = 20\cdot01$. So that there would be between 2000 and 3000 entries in our table.

3. We know that $1\cdot01^{231} = 10$, so that in using the table based on 1.01 we had frequently to add or subtract 231. It may be shown that $1\cdot001^{2304} = 10$, so that with a table based on 1.001, it would be necessary frequently to add or subtract 2,304. Can you suggest any means of simplifying this part of the work? Can you suggest another base that would be simpler to work with than 1.001?—If we could find a number k such that $k^{1000} = 10$, we should have to add or subtract 1,000, an easier operation than with 231 or 2,304.

4. Expand $(1+x)^{1000}$ in a series, and by substituting different values of x find a value that will make $(1+x)^{1000}$ equal to 10.—The series is

$$1 + 1000x + \frac{1000 \times 999}{2}x^2 + \frac{1000 \times 999 \times 998}{2 \times 3}x^3 + \dots$$

We have just found that when $x = 0\cdot001$ the series has the value 2.7, so that this value is too small. Trying 0.002, 0.003, ..., we find $1\cdot002^{1000} = 7\cdot4$ and $1\cdot003^{1000} = 20\cdot0$, so that x must lie between 0.002 and 0.003.

You could in the same way try 0.0021, 0.0022, 0.0023 as values of x , but the work is not especially instructive, and you may take it on trust that $1\cdot0023^{1000} = 10$ pretty exactly. In what other ways can you express this relation?—1.0023 multiplied by itself 1,000 times makes 10; or 1.0023 is the thousandth root of 10.

5. The following is an extract from the table calculated on 1.0023 as base:

Index	Power	Logarithm
1	1.0023	0.0009977
2	1.0046	0.0019982
3	1.0069	0.0029863
10	1.0232	0.0099605
20	1.0470	0.0199467
50	1.1217	0.0498767
100	1.2583	0.0997842.

The numbers 1·0023, ... have also been looked up in a table of logarithms, and the logarithm there given for each has been entered in the third column. Compare the first and third columns. Each entry in the last column multiplied by 1,000 is nearly equal to the corresponding entry in the first column. Hence, instead of calculating our table we can make use of a table of logarithms.

For instance, let us calculate $7\cdot76 \times 1\cdot58$. The extract given above is from a seven-figure table of logarithms; but for the present calculation and for most practical purposes a four-figure table is enough. We look up the number 7·76 in a four-figure table and find 0·8899 given as its logarithm. This tells us that $7\cdot76 = 1\cdot0023^{890}$. As the logarithm of 1·58 we find 0·1987, so we know that $1\cdot58 = 1\cdot0023^{199}$. Hence $7\cdot76 \times 1\cdot58 = 1\cdot0023^{890} \times 1\cdot0023^{199} = 1\cdot0023^{1089} = 1\cdot0023^{1000} \times 1\cdot0023^{89} = 10 \times 1\cdot0023^{89}$. One thousandth of 89 is 0·089, and we want from the table a number whose logarithm is about 0·089. The table gives

Number	Logarithm
1·227	0·0888
1·228	0·0892,

and from either of these we get approximately $1\cdot0023^{89} = 1\cdot23$. So that $7\cdot76 \times 1\cdot58 = 12\cdot3$.

Calculate in the same way $7\cdot72 \div 1\cdot52$, $1\cdot616 \times 1\cdot088$, $1\cdot616 \div 1\cdot088$, 107×793 , $793 \div 10\cdot7$, $7\cdot40 \times 7\cdot57$, $1525 \div 780$.

Negative and Fractional Indices.

6. Let us now recall when we first met with indices, what we have used them for, and what we have learned about them. We first met indices in connexion with an old problem of horse-dealing (chap. IX, art. 15). In connexion with that problem we had to calculate the product of 32 2's or 2^{32} . State again in ordinary English, as well as in mathematical symbols, the method we adopted to lighten the calculation.—We separated the 32 2's into two batches of 16; in mathematical symbols, we took $(2^{16})^2$ instead of 2^{32} . Then we separated the 16 2's of each batch into two batches of 8; or in symbols we took $(2^8)^2$ instead of 2^{16} .

How will you write the whole product of 32 2's at this stage? When several pairs of brackets occur,

it makes the expression clearer to use different kinds of brackets.—The whole product of 32 2's is $(2^8)^2$ multiplied by itself, which we may write as $\{(2^8)^2\}^2$. The last step is to separate each batch of 8 2's into two batches of 4 2's, to put $(2^4)^2$ instead of 2^8 . A batch of 16 2's is the product of two batches of 8, and so is looked on as $(2^4)^2 \times (2^4)^2$; this we write more shortly as $\{(2^4)^2\}^2$. And the whole product of 32 2's, being the product of two batches of 16, is $\{(2^4)^2\}^2 \times \{(2^4)^2\}^2$, or $[\{(2^4)^2\}^2]^2$.

The next process is the calculation of this number. The statement in mathematical symbols of this calculation is

$$[\{(2^4)^2\}^2]^2 = [\{16^2\}^2]^2 = [260^2]^2 = 68,000^2 = 4,600,000,000.$$

State this in English without the use of mathematical symbols or language.—We take a batch of 4 2's and multiply it out; the product is 16. Putting 16 in place of 2^4 gives the second mathematical form used. The next step is to multiply together the two batches of 4 2's, that is, to multiply 2 16's together; the product is, to two significant figures, 260. This corresponds to the third mathematical form $[260^2]^2$. We have now to multiply 260 by itself; the product is 68,000. Finally, we multiply 68,000 by itself, and as the product of 32 2's we get 4,600 million.

State what operations the symbols $(10^3)^2$, $(10^2)^3$, 10^6 , $(3^2)^5$, $(3^5)^2$, 3^{10} direct you to carry out. Carry out the operations and account for the results you obtain.

7. Indices turned up again in connexion with the Morse Code and the Semaphore (chap. X, art. 1). There we had such relations as $7^{10} \times 7 = 7^{11}$ and $2^n \times 2 = 2^{n+1}$. State in English what these relations mean, justify them, and generalize them. State what operations each of the following expressions directs to be carried out, and carry them out:— $10^3 \times 10^2$, 10^5 , $3^4 \times 3^2$, $3^2 \times 3^3 \times 3^4$, 3^6 , 3^9 .

8. Work again through the discussion of indices in chap. XII on the Slide Rule, from article 13 onwards. We there found that so long as m and n are positive whole numbers

A. $a^m \times a^n = a^{m+n}$;

B. $a^m \div a^n = a^{m-n}$ when $m > n$,
and $= 1/a^{n-m}$ when $m < n$;

C. $(a^m)^n = a^{m \times n}$.

Once more state what operations each of these expressions directs to be carried out, and so justify the equations $A B C$.

What operations do the following expressions direct you to carry out? Carry out the operations, giving each value to two significant figures. When two expressions have the same value, explain why:—

$$\begin{array}{lll} (2 \times 3)^5, & 2^5 \times 3^5, & (3 \times 4)^2, \\ 3^2 \times 4^2, & (2 \div 3)^5, & 2^5 \div 3^5, \\ (3 \div 2)^5, & 3^5 \div 2^5, & (4 \div 3)^2, \\ 4^2 \div 3^2, & (3 \div 4)^2, & 3^2 \div 4^2. \end{array}$$

You find such relations as $(2 \times 3)^5 = 2^5 \times 3^5$ and $(3 \div 4)^2 = 3^2 \div 4^2$. Generalize these relations and justify them.—
More general forms are

$$D. (a \times b)^m = a^m \times b^m;$$

$$E. (a \div b)^m = a^m \div b^m.$$

The expression $(2 \times 3)^5$ directs us to multiply 2 and 3 together, and then to take 5 such products and multiply them together. So in the final product there are 5 3's and 5 2's. And the expression $2^5 \times 3^5$ means simply the product of 5 2's and 5 3's, so the two expressions are equal. Nor have the particular numbers 2, 3, 5 any influence on the reasoning; the index 5 must be a whole number, the numbers 2 and 3 could be fractional. So we know that, m being a whole number, $(a \times b)^m = a^m \times b^m$. The relation $(a \div b)^m = a^m \div b^m$ is justified for integral values of m in the same way.

9. In chapter XII we found meanings for such expressions as 1.01^1 , 1.01^0 , 1.01^{-13} , $1.01^{0.5}$. In the same way show what the following expressions should mean, and give their values to two significant figures:—

$$2^0, 2^{-3}, 3^1, 100^0, 100^1, 100^{0.5}, 10^{0.5}, 36^{0.5}, 4^{-2}, 8^{-3}, 1 \div 2^{-3}, 1 \div 4^{-2}, 2^0 \div 2^{-3}, 3^1 \times 2^0 \div 2^{-3}, 10 \times 10^{0.5}, 10 \div 10^{0.5}, 10^2 \times 10^{0.5}, 10^3 \div 10^{0.5}.$$

State again how we found a meaning for a negative integral index?—When $m = 17$, $n = 30$, the relation B gives $a^{17} \div a^{30} = 1 \div a^{13}$, but if we pay no regard to the original meaning of the relation B and write it in the form appropriate to the case $m > n$, we have $a^{17} \div a^{30} = a^{-13}$. We

used this to define the negative index -13 , and as far as we could discover this definition led to no inconsistency.

10. Now let us define any integral negative index $-p$ by the equation

$$a^{-p} = 1 \div a^p.$$

Using this definition put $m = 4$ and $n = -3$ in the three equations A , B , C of article 8, and see if the relations are then true. Give a also a numerical value if you like.—The relation A becomes $a^4 \times a^{-3} = a^1$, and this means $a \times a \times a \times a \div (a \times a \times a) = a$; which is true. B becomes $a^4 \div a^{-3} = a^{4+3}$, since to subtract -3 is to add 3 (chap. VI, art. 16). Now if a^4 divided by any quantity P is equal to a^7 , then a^4 is equal to a^7 multiplied by P , so that our relation is the same as $a^4 = a^7 \times a^{-3}$, which means

$$a \times a \times a \times a = a \times a \times a \times a \times a \times a \times a \div (a \times a \times a),$$

or written more concisely,

$$aaaa = aaaaaaa \div aaa,$$

which is true. When $a = 2$, this relation is $16 = 128 \div 8$, when $a = 0.1$ it is $0.0001 = 0.0000001 \div 0.001$. The relation C becomes $(a^4)^{-3} = a^{-12}$. By definition $(a^4)^{-3}$ means $1 \div (a^4 \times a^4 \times a^4)$, that is, $1 \div aaaa.aaaa.aaaa$ or $1 \div a^{12}$ which is also written a^{-12} . When $a = 2$,

$$a^4 = 16, (a^4)^{-3} = 16^{-3} = 1/4096 = 0.00024,$$

and

$$a^{-12} = 2^{-12} = 1/2^{12} = 0.00024.$$

When $a = 0.2$,

$$a^4 = 0.0016, (a^4)^{-3} = 0.0016^{-3} = 1/0.0016^3 = 244,000,000.$$

Put $m = -3$ and $n = 4$ and $a = 2$ in the five relations A , B , C , D , E , and see if they are then true. Repeat with $a = 0.2$ and with $a = 4$.

Does the verification of these formulas for the negative index -3 depend on the particular number 3 chosen? —The reasoning is just the same for any other number.

We will then deal with negative indices on the definition at the beginning of this article and on the supposition that they obey the laws A , B , C , D , E , unless and until at any time we meet a case of inconsistency.

11. We gave a meaning also to one fractional index $1/2$ or 0.5 . It was agreed that $a^{0.5}$ was to be defined by the equation $a^{0.5} \times a^{0.5} = a$, or in other words, that $a^{0.5}$ was to mean the square root of a . Can you find a meaning for $a^{0.001}$ by assuming $a^{0.001}$ to obey the laws *A, B, C, D, E*?—*A* gives $a^{0.001} \times a^{0.001} = a^{0.002}$. This merely gives us a new index 0.002 for which we have no meaning. But if we take more factors, if we take 1,000 factors, we get $a^{0.001} \times a^{0.001} \times \dots \times a^{0.001} = a^1$. If this is to hold $a^{0.001}$ must be a number which, multiplied 1,000 times by itself, makes a ; or in other words, $a^{0.001}$ is the 1,000th root of a . For instance, $10^{0.001} = 1.0023$, for we know that 1.0023 multiplied 1,000 times by itself makes 10. Again, the relation *C* gives $(a^{0.001})^{1000} = a$, which is another way of stating the relation we have just dealt with.

What meanings will you give to $a^{0.002}$, $a^{0.003}$, $a^{0.004}$, $a^{0.074}$, $a^{0.341}$, $a^{4.824}$?—We have met $a^{0.002}$ already in the relation $a^{0.001} \times a^{0.001} = a^{0.002}$. We will use this, which may also be written $(a^{0.001})^2 = a^{0.002}$, as the definition of $a^{0.002}$. In the same way $a^{0.003}$ may be defined as $a^{0.001} \times a^{0.001} \times a^{0.001}$, or $(a^{0.001})^3$, and so for $a^{0.004}$. And in the same way $a^{0.074}$ is the product of 74 factors, $a^{0.001}$; $a^{0.341}$ the product of 341 factors; and $a^{4.824}$ the product of 4,824 factors.

Find the values of these numbers when $a = 10$, that is, the values of $10^{0.002}$, $10^{0.003}$, $10^{0.004}$, $10^{0.074}$, $10^{0.341}$, $10^{4.824}$.

12. Do the laws *A, B, C, D, E* hold for all the indices now defined, that is, for all fractions with 1,000 as denominator?—Suppose $m = 0.03$ and $n = 0.152$. Then *A* becomes $a^{0.03} \times a^{0.152} = a^{0.182}$. If for shortness we write k for $a^{0.001}$, this is $k^{30} \times k^{152} = k^{182}$; which in this form is obviously true. *B* gives $a^{0.03} \div a^{0.152} = a^{-0.122}$ or $1 \div a^{0.122}$, that is, $k^{30} \div k^{152} = k^{-122}$ or $1 \div k^{122}$, clearly true; this introduces the negative fractional index -0.122 and the definition of $a^{-0.122}$ as $1 \div a^{0.122}$. *C* gives $(a^{0.03})^{0.152} = a^{0.00456}$.

This introduces a new fraction not among those considered. We shall return to it. Meantime, is it true that $(a^{0.03})^{1000} = a^{30}$? More generally, is it true that $(a^{p/1000})^{1000} = a^p$?— $a^{p/1000}$ means $a^{0.001}$ multiplied by itself p times, so that $(a^{p/1000})^{1000}$ means $a^{0.001}$ multiplied by itself $p \times 1000$ times. We may arrange these $1000 \times p$ factors in batches of 1000.

Each batch is then $(a^{0.001})^{1000}$ or a ; so that the whole 1000 p factors make a^p , and so $(a^{p/1000})^{1000} = a^p$. More generally $(a^{p/q})^q = a^p$, and we could have used this as the definition of $a^{p/q}$.

13. Let us include all possible fractions among our indices by defining $a^{1/q}$ by the equation $(a^{1/q})^q = a$; and by defining $a^{p/q}$ by this combined with $(a^{1/q})^p = a^{p/q}$ or by $(a^{p/q})^q = a^p$, which two definitions we have seen to be equivalent.

This gives two definitions of $a^{0.03}$ according as we take 0.03 as $3 \div 100$ or $30 \div 1,000$. Are these two definitions equivalent?—The question is whether $(a^{3/100})^{100} = a^3$ and $(a^{30/1000})^{1000} = a^{30}$ are equivalent. Let us denote the 100th root of a by p , and the 1000th root by q . Now 1000 q 's multiplied together make a , and we can arrange the 1000 q 's in 100 batches of 10. The product of the 100 batches is a , so that each batch is the 100th root of a , that is $q^{10} = p$. The first relation above is $(p^{3/100})^{100} = a^3$ or $(p^{100/3})^3 = a^3$ or $p^{100} = a$. The second is $(q^{30/1000})^{1000} = a^{30}$ or $(q^{1000/30})^{30} = a^{30}$ or $q^{1000} = a$. And we know $q^{10} = p$, so that $q^{1000} = p^{100}$, and the two definitions are equivalent.

The 100th root of 10 is 1.0233 and the 1000th root is 1.0023. Calculate independently 1.0233³ and 1.0023³⁰.

Consider $(a^{0.03})^{0.152} = a^{0.00456}$ as an example of the more general $(a^{p \div q})^{r \div s} = a^{p \times r \div q \div s}$.—We write P for $a^{0.03}$, and Q for $P^{0.152}$. By definition $Q^{1000} = (P^{0.152})^{1000} = P^{152}$. So that $Q^{1000 \times 100} = (P^{152})^{100} = (P^{100})^{152} = (a^3)^{152} = a^{3 \times 152}$ or $Q^{100000} = a^{456}$. But $(a^{0.00456})^{100000} = a^{456}$ is the definition of $a^{0.00456}$.

State what operations each expression in the preceding paragraph directs to be carried out.

14. Any number, integral or fractional, positive or negative, has now a meaning as an index. The base alone is restricted to a positive number, integral or fractional. The arrangement of a table of logarithms can now be explained. It is a series of indices and corresponding powers, but the base is not 1.0023 although we were able to use the table for that base. The base is 10 and the relation between the index x and the power y is $y = 10^x$. This relation is also spoken of as that between a number y and its logarithm x , and is written in the form $x = \log y$.

In article 5 we found that the table gave 0.0009977 as the logarithm of 1.0023. This we write either as $10^{0.0009977} = 1.0023$ or as $0.0009977 = \log 1.0023$. To see whether this entry tallies with our definition of a fractional index we should have to calculate 10^{9977} and $1.0023^{10,000,000}$ and see if they are equal. This is laborious, and we can find easier ways of checking the table.

15. As a first check on the table take 0.001 as an approximation to 0.0009977.—

The relation to be checked is then $10^{0.001} = 1.0023$ or $10 = 1.0023^{1000}$, which we already know to be true.

Again, by a geometrical construction find the square root of 10.—We draw a circle on a diameter 11 units long (Fig. 1), and divide the diameter into parts a of length 10 and b of length 1. At the point of division we draw the

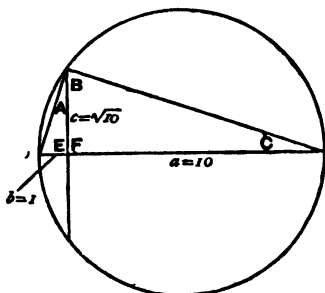


FIG. 1.

⊥ chord. Half its length c is $\sqrt{10}$. The angles A and C of the figure are equal, each being $90^\circ - B$, and the angles F and F are equal, each being right. So that we have two

similar triangles and $\frac{a}{c} = \frac{c}{b}$ or $c \times c = a \times b$ or $c = \sqrt{a \times b} = \sqrt{10}$. We could also begin with a diameter of length 7 divided into parts of 5 and 2. This will give a better result. The size of the figure is also important for accuracy. The square root of 10 is 3.16.

Use also your graph of $y = 1.01^x$ (chap. XII, art. 8) to find the square root of 10.

We have thus $10^{0.5} = 3.16$ or $0.5 = \log 3.16$. Compare the entry in the table of logarithms.

16. Use the same methods to find the square root of 3.16.—It is 1.78. What is its relation to 10?— $1.78^2 = 3.16$, so that $1.78^4 = 3.16^2 = 10$, or $1.78 = 10^{0.25}$, or $\log 1.78 = 0.25$.

Continue by extracting the square root of 1.78, and then

the root of *this* root, and so on. Tabulate your results and compare them with the logarithm tables. The results are:

Number	Logarithm
10	1
8.16	$1/2$ or 0.5
1.78	$1/4$ or 0.25
1.33	$1/8$ or 0.125
1.15	$1/16$ or 0.0625
1.07	$1/32$ or 0.0312
1.04	$1/64$ or 0.0156.

Now calculate the entry for the index or logarithm $3/64$ or 0.0468. — $10^{3/64} = 10^{1/32 + 1/64} = 10^{1/32} \times 10^{1/64} = 1.07 \times 1.04 = 1.11$.

In this way all the entries could be found for which the logarithm is a multiple of $1/64$.

17. By continuing these processes of root-extraction and calculation we could make a table of as many entries as we please. But the errors of drawing and measuring would soon accumulate till the results were useless. An accurate but heavy method of constructing a table would be to calculate the successive square roots by the ordinary arithmetical process. The methods actually in use for calculating the tables are much more expeditious; we shall not discuss them here.

18. We have now in the logarithm table a means of calculating to a considerable degree of accuracy. Use it to calculate to 4 significant figures the quantities in chapter XII, art. 6, namely, 7.76×1.58 , 7.72×1.52 , 1.616×1.088 , $1.616 \div 1.088$, 107×793 , $793 \div 10.7$, 7.40×7.57 , $1525 \div 780$, and see how far out our previous calculations of these quantities were. — The first is $7.76 \times 1.58 = 10^{0.8899} \times 10^{0.1987} = 10^{1.0886} = 10 \times 10^{0.0886} = 10 \times 1.226 = 12.26$.

With all the steps made separately this is

$$7.76 = 10^{0.8899}$$

$$1.58 = 10^{0.1937}$$

$$0.8899 + 0.1987 = 1.0886$$

$$1.0886 = 1 + 0.0886$$

$$10^1 = 10$$

$$10^{0.0886} = 1.226$$

therefore $7.76 \times 1.58 = 10 \times 1.226$.

Rewrite this, using the logarithm notation instead of the index notation.—It is

$$\log 7.76 = 0.8899$$

$$\log 1.58 = 0.1987$$

$$0.8899 + 0.1987 = 1 + 0.0886$$

$$1 = \log 10$$

$$0.0886 = \log 1.226$$

therefore $7.76 \times 1.58 = 10 \times 1.226.$

19. State the equation $7.76 = 10^{0.8899}$ in terms of logarithms.—0.8899 is the logarithm of 7.76 (to the base 10), or shortly $0.8899 = \log 7.76$. Express 0.1987 and 1.0886 similarly, and then express the equation $0.8899 + 0.1987 = 1.0886$ in terms of 7.76, 1.58, and 12.26.— $0.1987 = \log 1.58$, $1.0886 = \log 12.26$; $\log 7.76 + \log 1.58 = \log 12.26$. And if you substitute for 12.26 its value in terms of 7.76 and 1.58?—The relation is

$$\log 7.76 + \log 1.58 = \log (7.76 \times 1.58).$$

Calculate 7.72×1.52 in the same way, and express the result in various ways.—The product is 11.73, and 7.72 being $= 10^{0.8876}$, &c., the equation $10^{0.8876} \times 10^{0.1818} = 10^{1.0694}$ is simply another way of writing $7.72 \times 1.52 = 11.73$. Again, we state the relations as $0.8876 = \log 7.72$, &c., and the equation $0.8876 + 0.1818 = 1.0694$ may be written $\log 7.72 + \log 1.52 = \log 11.73$, or $= \log (7.72 \times 1.52)$.

The calculation of 7.72×1.52 is usually more compactly written in such a form as:—

$$\log 7.72 = 0.8876$$

$$\log 1.52 = 0.1818$$

$$1.0694 = \log 11.73$$

$$7.72 \times 1.52 = 11.73.$$

Proceed with the remaining calculations.

20. We have had several equations like

$$\log 7.76 + \log 1.58 = \log (7.76 \times 1.58).$$

Give a general form for such equations and justify it as the

equation just written was justified.—The general equation is

$$\log a + \log b = \log (a \times b).$$

To justify it we suppose a number p found such that $a = 10^p$, or (what is the same thing) $p = \log a$; and a number q such that $b = 10^q$ or $q = \log b$. Then $a \times b$ is $10^p \times 10^q$ or 10^{p+q} , and another way of expressing this relation is $p + q = \log (a \times b)$; and since p means $\log a$ and q means $\log b$, the equation we began with is justified.

21. Some time ago (chap. VI), we discussed what the height and diameter of a standard gallon measure must be. Such a measure has its diameter and height equal, and has a capacity of 277.4 cubic inches. Can you use logarithm tables to give these dimensions?—If d is the diameter in

inches we know that $\frac{\pi}{4} \times d \times d \times d = 277.4$,

$$\text{or} \quad d \times d \times d = \frac{277.4 \times 4}{3.142} = \frac{2.774 \times 4 \times 100}{3.142}.$$

$$\log 2.774 = 0.4431$$

$$\log 4 = 0.6021$$

$$\log 100 = 2$$

$$\hline 3.0452$$

$$\log 3.142 = 0.4972$$

$$\hline 2.5480$$

$$2 = \log 100$$

$$0.5480 = \log 3.532$$

so that

$$d \times d \times d = 3.532$$

and therefore $\log d + \log d + \log d = \log 3.532$.

The left side of this equation is $3 \times \log d$, and we have seen that $\log 3.532 = 0.5480$, so that

$$3 \times \log d = 0.5480$$

or

$$\log d = 0.1827$$

and the table gives

$$0.1827 = \log 1.52$$

so that

$$d = 1.52,$$

and both diameter and height of the gallon measure are 7·068 inches.

Ex. 1. Calculate the height and diameter of a standard litre measure whose capacity is 1000 c.cm., and whose height is twice its diameter.

22. Compare the method of extracting a cube root by logarithm tables with that by means of the graph of x^3 (chapter VI), and with the ordinary arithmetical method. The graph method is only a rough method. The arithmetical extraction of a square root is cumbersome, for a cube root it is forbidding, and for higher roots the labour is so great that to calculate such a number as $2\cdot718^{3\cdot142}$ without logarithm tables is possibly beyond the patience of any man. Logarithm tables are a great help for multiplication and division, but their true value as a labour-saving device first appears in root extraction.

Ex. 2. Calculate the number $2\cdot718^{3\cdot142}$.

23. Can you generalize the relation

$$\log(d \times d \times d) = \log d + \log d + \log d,$$

or (more compactly) $\log d^3 = 3 \log d$?—If n is any whole number $\log d^n = n \log d$, for this is only the short way of writing $\log(d \times d \times d \dots) = \log d + \log d + \dots$, there being n factors d on the left, and n terms $\log d$ on the right.

Can you deduce this relation from the meaning of logarithm?—Suppose $\log d = k$, then $d = 10^k$, and $d^3 = (10^k)^3 = 10^{3k}$, so that $\log d^3 = 3k$, that is, $= 3 \log d$. And so for the case when 3 is replaced by n .

Is it true that $\log d^n = n \log d$ if n is fractional?—Put $\log d = k$, then $d = 10^k$ and no matter whether n is integral or fractional $d^n = (10^k)^n = 10^{kn}$, so that $\log d^n = nk = n \log d$.

24. The quantities z and u being related by $z = 10^u$, look out a number of corresponding values in the tables and draw the graph of z as a function of u from $z = 1$ to $z = 10$. What are the corresponding limits for u ? It was stated (in art. 1 of this chapter) that the graphs $y = 1\cdot01^x$ and $y = 1\cdot001^x$ differed only in the graduation of the x -axis. Can you choose such a scale for u that the curve $z = 10^u$

will fit the curve $y = 1.01^x$.— u runs from 0 to 1, and x runs from 0 to 231, so let us try making $x = 231 \times u$ for all values of x and u . Then for any value of x $y = 1.01^x = 1.01^{231u} = (1.01^{231})^u$; and since $1.01^{231} = 10$ this is $= 10^u$. Therefore $y = z$; and to make $x = 231u$ for all values of x and u does make the graphs fit.

25. Can you explain why our logarithm table was useful to give x from y or y from x when $y = 1.0023^x$? That is, how would you make the curve $z = 10^u$ fit the curve $y = 1.0023^x$?— x runs from 0 to 1000, u from 0 to 1. If $x = 1000u$ for all values of u , then $y = 1.0023^{1000u} = (1.0023^{1000})^u$, and this is $= 10^u$ since $1.0023^{1000} = 10$. To get x from the table we multiplied the logarithm there given by 1000, so that the relation $x = 1000u$ expresses exactly what we did.

Ex. 3. Draw the graph of $z = 10^u$, not only from $u = 0$ to $u = 1$, but for as great a range of values of u as possible.

EXERCISES.

4. In t seconds a body falls $16t^2$ feet. How many feet does it fall in 5 seconds, and how many seconds does it take to fall 100 feet?

5. From a cylinder of wood with a diameter of 11 inches a rectangular beam is cut. If the breadth of the beam is 8 inches, find the depth from the formula $d = \sqrt{D^2 - b^2}$, where b is the breadth and d the depth of the beam and D the diameter of the cylinder.

If the strength of a beam varies as bd^2 , compare the strengths of beams 3, 4, 5, 6, 7, 8 inches broad cut from a cylinder 11 inches in diameter, and so find, roughly, the breadth of the strongest beam which can be cut from the cylinder.

6. The Board of Trade rule for flat stayed surfaces of marine boilers is $p = \frac{c(16t+1)^2}{s-6}$, where p is the safe working pressure in lbs. per square inch, t is the thickness of the plate in inches, and s is the area of plate supported by each stay in square inches. The 'pitch' of the stays, a , is given by $t = 0.009a\sqrt{p}$ for steel plates. Assume that $s = 1.5a^2$, $c = 60$, and find the safe working pressure which may be used with plates 1 inch thick.

7. The height of a pint pot of a certain shape is 12.6 cm. to the nearest tenth. A litre being 1.76 pints to the nearest hundredth,

find the heights of pots of the same shape that will hold 3 decilitres, 5 decilitres, and 1 litre.

8. The population of the United Kingdom was estimated at 37,802,000 in the middle of the year 1891, and at 41,551,000 ten years later. Assuming that population increases (i) by the same number of individuals every year, (ii) by a constant percentage every year, estimate the population in the middle of the years 1894 and 1897.

9. The earth completes a revolution round the sun in 365 days, and the planet Venus completes a revolution in 225 days. The distance from the earth to the sun is 90 million miles. Find the distance of Venus from the sun, assuming that the square of the time of a planet's revolution is proportional to the cube of its distance from the sun.

10. Obtain the quotients

$$\frac{(y^2 - 3y + 2) - (x^2 - 3x + 2)}{y - x} \quad \text{and} \quad \frac{y^5 - x^5}{y - x}.$$

The limiting values of these quotients when y is put equal to x after division are called the differential coefficients of $x^2 - 3x + 2$ and x^5 respectively. Show that the differential coefficient of $x^2 - 3x + 2$ is $2x - 3$ and that of x^5 is $5x^4$.

11. A ladder 7.9 metres long is placed against a wall, the foot of the ladder being 3.8 metres from the bottom of the wall. If the height of the point on the wall reached by the ladder is x metres, find x , given that $x^2 + 3.8^2 = 7.9^2$.

12. The area of the surface of the human body is proportional to the square of the height, and the weight is proportional to the cube of the height. The loss of heat by the body varies as the surface area, and this loss must be made up by food. Does a man of 150 lbs. or his son of 110 lbs. need more food for this purpose in proportion to his weight? If the father needs 7 ounces a day for this purpose, what does the son need?

13. Given that the area of any triangle the lengths of whose sides are a , b , and c is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$, reduce this to its simplest form for a triangle in which $a^2 = b^2 + c^2$, and calculate the area when $b = 6.78$ inches, $c = 4.39$ inches.

14. The gas-service pipe to a house 75 feet from the main is $\frac{7}{8}$ inch in diameter. For how many burners, each taking 5 cubic feet of gas per hour, will this serve? The number of cubic feet

per hour delivered by a pipe on that main is $1000 \sqrt{\frac{d^5}{0.45L}}$, where

d is the diameter of the pipe in inches, and L is the length of the pipe in yards.

15. A penny measures 1·23 inches across and a halfpenny 1 inch. If the thicknesses were in the same ratio, what fraction (as a decimal to the nearest hundredth) would the weight of the halfpenny be of the weight of the penny?

If a penny weighed twice as much as a halfpenny, and was of the same thickness, what would it measure across (to the nearest hundredth of an inch), the halfpenny measuring 1 inch?

If a disc measures a inches across and b inches in thickness, then its volume is $0\cdot7854 \times a \times a \times b$ cubic inches.

16. The time of swing of a pendulum is proportional to the square root of its length. If a pendulum 39 inches long swings once per second, how many swings per minute will be made by a pendulum 5 feet long?

17. On a map drawn on a scale of an inch to a mile two points are found to be 1·3 inches apart. One point is at the sea level, the other at a height of 1,720 feet. Lay off on squared paper to any suitable scale the horizontal and vertical distances between these points, and by measurement find the length of wire that would run straight between these points.

Having given that

$$(\text{distance between the points})^2 = (\text{horizontal distance})^2 + (\text{vertical distance})^2,$$

calculate the length of wire in feet to the nearest hundred.

18. In return for an immediate payment of £575 17s. 6d. a Life Assurance Society undertakes to pay £1,000 at my death, reckoning that on the average a man of my age will live nineteen years more. Find, to the nearest tenth per cent, what rate of interest the Society pays on the average for money thus deposited with it, assuming that a sum of £ p invested at r per cent. amounts in n years to £ $p(1+r/100)^n$.

Investigate the correctness of the formula used.

19. Find the value of $(a^{1/5} + b^{1/5})^{1/10} \div c^{1/4}$, where $a = 1009\cdot6$, $b = 2419\cdot3$, $c = 5769$.

20. Explain the meaning of a^2 and $a^{1/5}$.

Between what integers does $87^{1/5}$ lie? Give reasons for your answer.

21. The formula $p = P \div 2\cdot718^a/28000$ gives the pressure p of the atmosphere in pounds per square inch at a height of a feet above sea level, P being the pressure in pounds per square inch at the sea level. Find the pressure at the top of Snowdon, 3570 feet above sea level, when $P = 15$.

22. Give an account in continuous English of indices, explaining the convenience of the notation for the integral index, and the extension from integral to negative and fractional indices. Importance is attached to the form as well as the matter of the account.

23. A jet of water issues from the side of a tank with a horizontal velocity of $\sqrt{2gH}$ feet per second, where H feet is the head of water above the orifice, and g a certain constant. The water retains this horizontal velocity unchanged while falling, and in t seconds from issuing it falls $0.5gt^2$ feet. Taking $H = 10$, and $g = 32$, find whereabouts the jet will strike a horizontal plane 100 feet below the orifice.

24. The relation between p and v indicated by the table

$v =$	4.6	5.2	5.8	6.4
$p =$	73	63	55	

is suspected to be given by the equation $pv^x = k$, where x and k are constants. Allowing for slight errors of observation, find, as nearly as you can, the values of x and k , and give the value of p corresponding to the value 6.4 of v .

25. If $a^n = b^y$, and $b = a^c$, show that $y = x/c$. With the aid of the tables of logarithms apply this result to find the values of $\log_e 5$ and $\log_e 211$, where $e = 2.718$.

26. It is said that any angle ABC can be trisected by the following method:

'On tracing-paper draw a circle, centre O . Produce any radius OD to E so that $DE = OD$, and through D draw MN of indefinite length perpendicular to OE . Place the tracing-paper figure so that the whole of the circle lies within the angle ABC , arrange it so that MN passes through B , one arm of the angle passes through E and the other touches the circle. Then prick through D and O , and join the points thus obtained to B .'

Carry out this construction for an angle of 80° , using a circle with a radius of 3 cm.; test the result with your protractor, and either justify or disprove the soundness of the method.

27. The weight of the model of a bell made of the same metal as the bell on the scale of 5 centimetres to the metre is 26 grams. What is the weight of the bell?

28. Draw a circle of radius 6.6 cm. Draw a diameter AB and produce it to O so that $BO = 3.2$ cm. Suppose a line originally lying along CBA to turn about O till it ceases to cut the circle. Call P and Q its points of intersection with the circle. Make a table

like the following. Draw OPQ in the positions named, and fill up the table :—

Position	Length of CP in cm.	Length of CQ in cm.	Area of rectangle $CP.CQ$ in sq. cm.
When $\angle ACQ = 0^\circ$ -			
" " 10° -			
" " 20° -			
" " 30° -			
" " 40° -			
When CPQ is just ceasing to cut the circle -			

: State the property suggested by the last column. Can you explain why the numbers in the last column are not exactly equal?

29. On squared paper draw two straight lines AB and AC , containing an angle of 35° . Draw in several positions the base of a triangle, of which two sides lie along AB and AC , and whose area is 4 square inches, and so draw the curve that touches the base in all positions.

Draw also the curve that touches the base in all positions, when the two sides lie along AB and AC , and the length of the perpendicular from A on the base is 2.3 inches.

By laying your ruler against the two curves draw the base of the triangle that has an area of 4 square inches, and has also the perpendicular from A on the base 2.3 inches long.

Also state how you could make a triangle having a vertical angle of 35° , area 4 square inches, and perpendicular from vertex on base 2.3 inches, without drawing any other loci than straight lines and circles.

30. About 3 inches away from the lower edge of a page draw a horizontal line MN right across the page, and from about its middle point X erect a perpendicular XF 2 cm. long. On the tracing-paper draw 10 circles having the same centre, the radii being 1, 2, 3, 4, &c., cm. respectively; then place the tracing-paper so that the circumference of the smallest circle passes through F and touches MN , and in this position prick through the centre of the circle. Deal with the second and the remaining circles in a similar way, determining the positions of the centre points on both sides of FX . Draw a freehand curve through the points thus obtained.

31. Draw a straight line to represent a coast, and two points L and M on it to represent lighthouses. Mark a point R to repre-

sent a rock in the offing. Measure the angle LRM . If a ship S is sailing past, prove that she will certainly be out of danger from the rock if she is steered so that the angle LSM is always less than the angle LRM .

Draw the course of the ship, supposing the angle LSM is constantly kept at 125° . State your method.

32. Once in ancient Greek times the problem of making a cubical altar of double the volume of an existing cubical altar had great importance. Take the edge of the original altar as 80 cm. and solve it (i) in the Greek fashion by the intersection of the graphs $y^2 = 160x$ and $x^2 = 80y$, (ii) by the graph $y = x^3$, (iii) by logarithms, and (iv) by the solution of the equation $x^3 = 2$ as follows:—This equation has a root between 1 and 2; by putting $x = 1 + y$ find from it another having its roots less by 1. By putting $y = z/10$ find an equation in z . This equation has a root between 2 and 3; find from it an equation in u with its roots less by 2, and so on.

CHAPTER XV

THEOREM OF PYTHAGORAS

1. NAME any case you have had of a right-angled triangle in which you knew a relation between the three sides.—

If C is the centre of a circle, AB a diameter, and MO the $\perp$ let fall on AB from a point M of the circumference, we know that $CM^2 = CO^2 + OM^2$ (chap. XIII, art. 19). Also if C is the centre of a circle and OG is a tangent to the circle at G we know that $OC^2 = OG^2 + GC^2$ (chap. XIII, art. 23). In each case what condition must be satisfied in order to make such a relation true?—It is enough to know that the triangle is right-angled. For suppose ABC a triangle in which B is

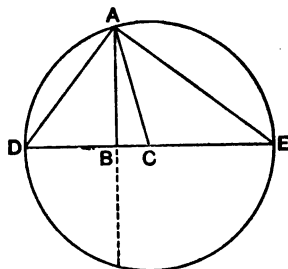


FIG. 1.

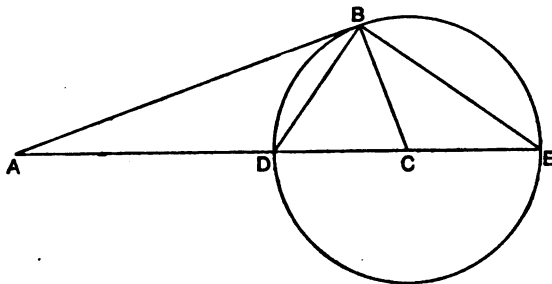


FIG. 2.

a right angle (Fig. 1). With C as centre draw a circle passing through A , and produce BC to form a diameter

DE of this circle; for such a figure it has been proved that $AC^2 = AB^2 + BC^2$. Or, again, with C as centre draw a circle through B (Fig. 2), and produce AC to form a diameter DE of this circle; we know that for this figure $AC^2 = AB^2 + BC^2$. Repeat these two proofs.

Ex. 1. Redraw the triangle ABC on squared paper, and draw the squares on AB , BC , and AC . Count the number of little squares contained in each of these three squares, and compare with the previous result.

Ex. 2. With 1 cm. or 2 cm. as unit draw a triangle ABC having $\angle B = 90^\circ$, $BA = 3$ units, $BC = 4$ units. Find the length of AC by measuring, and also by calculation. (This property of the numbers 3, 4, 5 has been known from very early times, and is still used by builders to square the foundations of a house.)

2. Draw on squared paper a triangle ABC (Fig. 8) having $\angle B = 90^\circ$, $a = 4$, $c = 3$. Draw the squares on the sides of the triangle, and count the number of unit squares included in each. In the case of the square $ACDE$ on b there are fractions to take account of. Can you avoid the discussion of these fractions by considering where D and E lie?—We go from A to E by going 4 units in the direction BA and 3 in the direction BC ; let us say 4 units east and 3 units south. From C to D is also 4 units east and 3 south.

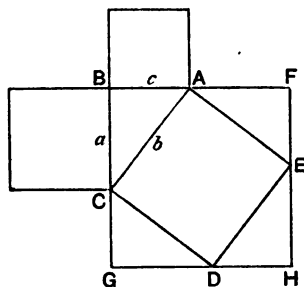


FIG. 3.

Prove that if D and E are found in this way $ACDE$ is a square.— $\triangle ABC$ and EFA have $\angle B = \angle F$, $BC = FA$, $BA = FE$, so that the $\triangle$ s are congruent and $AC = AE$. The congruence gives also $\angle BCA = \angle FAE$, so that $\angle FAE + \angle BAC = \angle BCA + \angle BAC = 90^\circ$, leaving $\angle CAE$ also $= 90^\circ$. In the same way $CD = CA$ and $\angle ACD = 90^\circ$. Thus CD is $\parallel$ to AE and the figure is a parallelogram. Thus all the sides are equal and all the angles right. What is the area of the square $BFHG$ of Fig. 3?— BF is 7 units long and

the square contains 49 unit squares. Can you express the square on b by means of it?—We must subtract the 4 triangular corners, each equal to $\triangle ABC$ or six unit squares. So that b^2 is $49 - 24$ or 25 unit squares.

3. Generalize the preceding discussion.—If BC is a units long and BA c units, the length of BF is $a + c$, and the area $BFHG$ is $(a + c)^2$ unit squares. And, the area of $\triangle ABC$ being $\frac{1}{2}ac$, we know that $b^2 = (a + c)^2 - 2ac$. Give a diagram showing the squares on a , c , and $a + c$, and the rectangle ac .—Fig. 4 shows that $(a + c)^2 - 2ac$ is $a^2 + c^2$; which we could have found algebraically by multiplying out $(a + c) \times (a + c)$.

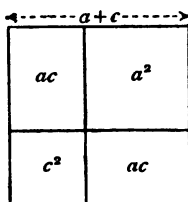


FIG. 4.

Again, discuss the case of Fig. 5, made by drawing lines $\parallel$ to a and c through the corners of the square $ACDE$.— GF is $a - c$ units, $GFKH$ is $(a - c)^2$ square units, and $b^2 = (a - c)^2 + 2ac$. Fig. 6 is composed of the square on $a - c$, and two rectangles each $= ac$. If instead we divide it by the dotted

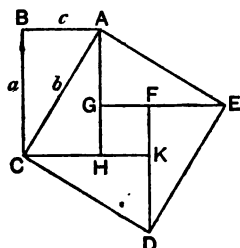


FIG. 5.

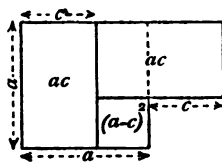


FIG. 6.

line, it is composed of the squares on a and c . So that $b^2 = a^2 + c^2$. This also we find by multiplying out $(a - c)^2$.

4. An ancient Indian geometer, to show how the two squares on the sides of a right-angled triangle may be cut up and fitted together so as to form the square on the hypo-

tenuse, gives such a figure as Fig. 7, and says concisely, 'Behold!' In modern Europe it is the custom to give formal statement and proof. Supply this formal statement and proof.

5. Take any right-angled triangle (Fig. 8) of which AB or c is the hypotenuse. Let it move at right angles to the side

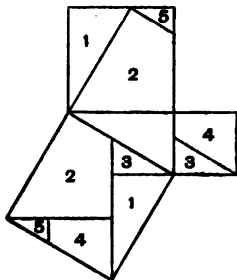


FIG. 7.

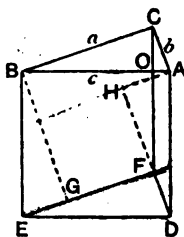


FIG. 8.

c through a distance c to the position DEF . During the motion the side c is continually covering up a new part of the paper while the sides a and b are continually uncovering parts previously covered. Since the area covered by the triangle is always the same, the total area covered up by c is equal to the total area uncovered by a and b . Give expressions for these areas, first ascertaining the $\perp$ distance between the two positions of a , and the $\perp$ distance between the two positions of b .—Congruence of Δs shows the $\perp$ distance BG to be a , and the $\perp$ distance AH to be b . So that side a traces out an area of a^2 , and side b an area of b^2 . Hence once more we have $c^2 = a^2 + b^2$.

Find the areas traced out by a and b by considering the $\perp$ distances (on Fig. 8) between BE and CF and between AD and CF .—If CF cuts AB in O the area traced out by a is $BE \times BO$, and the area traced out by b is $AD \times AO$.

Show how to divide the square on c into two rectangles equal to a^2 and b^2 .

6. Draw a right-angled triangle ABC with the 3 squares on its sides, as in Fig. 9. Draw $CLM \parallel$ to AD , and join KB

and CD . Compare as to area the $\triangle KAB$ and the square $KACH$ which have the same base and the same height. In the same way compare $\triangle CAD$ and rectangle $ADML$. If $\triangle KAB$ is rotated about A in the direction of the arrow through a right angle, how will it lie? Thus compare the areas $KACH$ and $ADML$, and in the same way compare the areas $BCGF$ and $BEML$.

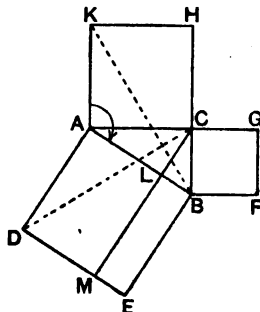


FIG. 9.

The discovery of the property of a right-angled triangle that the square on the hypotenuse is equal to the sum of the squares on the other two sides is ascribed to the Greek philosopher Pythagoras. The proof sketched in this article is given by the Greek geometer Euclid.

7. We have seen that if a $\triangle ABC$ has C a right angle, we can draw a circle with centre B and radius BA , produce BC to form a diameter DE , and produce AC to form a chord AF (Fig. 10), and so show that $b^2 = AC \cdot CF = DC \cdot CE = (c-a)(c+a) = c^2 - a^2$, so that $c^2 = a^2 + b^2$. Suppose now that we know that $c^2 = a^2 + b^2$, but do not know whether the angle C is right. Make the same construction, and see if you can tell whether $\angle C$ is right. —In this case we do not know whether CF (Fig. 10) is equal to CA . But we know that $CA \cdot CF = DC \cdot CE = (c-a)(c+a) = c^2 - a^2$. And we know that $c^2 = a^2 + b^2$, so that $CA \cdot CF = b^2$, and $CF = b^2/CA = b$. The chord AF is bisected at C , therefore BC is $\perp$ to ACF , and the angle C is right.

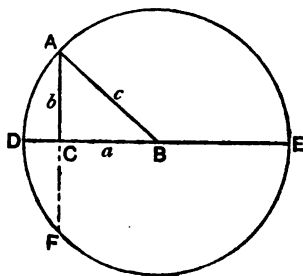


FIG. 10.

So that a triangle in which the square on one side

is equal to the sum of the squares on the other two is a right-angled triangle.

Determination of right-angled triangle from various conditions.

8. Draw any triangle ABC right-angled at C , and draw $CD \perp$ to AB (Fig. 11). Suppose BC to be of length a

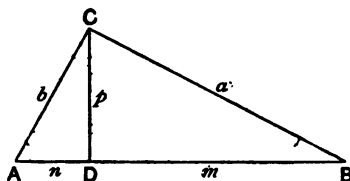


FIG. 11.

centimetres, and so on, as marked in the figure. How many of these lengths do you need to know to fix the triangle in size and shape? Will one length do, for example a ?—One length is not enough, for we can make any number of different triangles all of

which have the side a of the given length; and so for any other single length.

Will two lengths do?—They do in some cases at least, for example a and b .

In how many ways can you choose two quantities from the five a, b, p, m, n ?—The number of ways is $5 \times 4 \div 2$ or 10.

Justify this formula $5 \times 4 \div 2$. Try all these 10 ways.

9. When the two given sides are a and b , or p and n , or p and m , the method of drawing the triangle is obvious. When a and m are given how do you proceed?—We lay down BC of the given length a cm. And D is on the locus of points at which CB subtends a right angle, that is, on the circle on CB as diameter; D is also at distance m cm. from B . The intersection of the loci fixes its position. CA drawn $\perp$ to BC cuts BD produced in A . Name the similar cases given by other pairs of the quantities a, b, p, m, n .—Similar cases are given by the pairs a, p ; b, n ; b, p .

What cases remain to discuss?—The remaining cases are a, n ; b, m ; m, n . When m and n are given we lay down ADB , and C is given as the intersection of the $\perp$ at D and the circle on AB as diameter.

10. There remains to discuss the case in which a and n are given ; or the case in which b and m are given ; the two being essentially the same.

Suppose $a = 10.1$ and $n = 3.2$. Lay down BC and find the triangle by trial.—We erect a $\perp$ to BC at C (Fig. 12), and take A in succession at various points on the $\perp$. We join AB and let fall a $\perp$ CD from C . As A moves away from C the length AD increases ; and when

$AD = 3.2$ cm.

we have the triangle required. Repeat this with the help of tracing-paper.

What is the locus of D ? How can you simplify the preceding method by the help of this locus? And how by the help of this locus along with tracing-paper?

Again, begin by laying down $AD = 3.2$ cm.—We erect a $\perp$ at D (Fig. 13) and take C at various points on it in

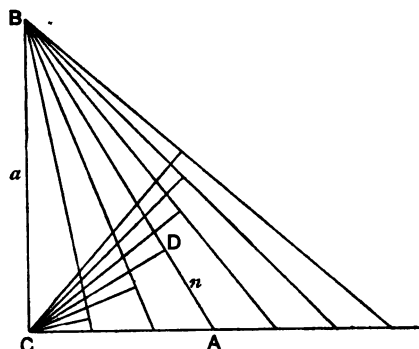


FIG. 12.

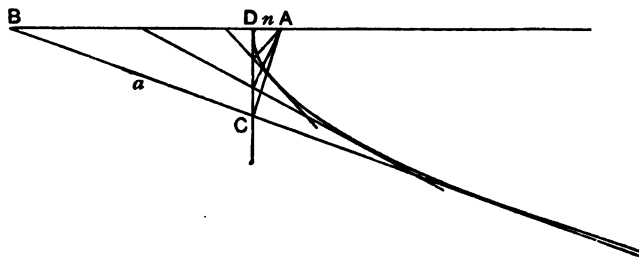


FIG. 13.

succession. We draw $CB \perp$ to CA meeting AD produced in B . As C moves away from D CB increases in length. The position that makes $CB = 10.1$ cm. gives the triangle we

want. Take C in a great number of positions. In each case produce BC , and draw the curve that all these positions of BC touch.

Begin again with AD , and find the triangle ABC by the help of tracing-paper.

11. Can you, by considering similar triangles or the rectangle property of a circle, give a relation between a , n , and any third length about our right-angled triangle ABC (Fig. 11)?—The triangles BCD and BAC are similar, so that $\frac{a}{m} = \frac{m+n}{a}$. The circle about ADC has its centre on AC , so that BC is a tangent and $a^2 = m(m+n)$. Give n its value $3\cdot2$, draw the graph of $m(m+3\cdot2)$, and so find the value of m that makes this expression equal to a^2 , that is, $= 102$. Use this value of m along with $n = 3\cdot2$ to draw the triangle, and measure a on the triangle. Also use this value of m along with $a = 10\cdot1$ to draw the triangle and measure n .

Multiply out $(m+1\cdot6)^2$.—It is $m^2 + 3\cdot2m + 2\cdot56$. If you denote $m+1\cdot6$ by x , can you give an equation for x corresponding to the equation $m(m+3\cdot2) = 102$ for m ?—Since $x^2 = m^2 + 3\cdot2m + 2\cdot56$, we know that $x^2 = 102 + 2\cdot56 = 105$. Now calculate x and m .—Logarithm tables give $\log 105 = 2\cdot0212$, so that $\log x = 1\cdot0106$, and $x = 10\cdot2$. Hence $m+1\cdot6 = 10\cdot2$ and $m = 8\cdot6$.

Calculate m for right-angled triangles that have

- i $a = 14\cdot2, \quad n = 4\cdot7$;
- ii $a = 12\cdot0, \quad n = 6\cdot8$;
- iii $a = 18\cdot8, \quad n = 10\cdot0$;

draw the triangles and measure a on each. Is the triangle always possible whatever values of a and n are given?

12. We have seen how the value of m may be calculated from the equation $m^2 + 3\cdot2m = 102$. Of course m can only be calculated from $m^2 + mn = a^2$ when n and a are given numerical values. But possibly you can put the relation in such a form as to indicate how the calculation is to be carried out when n and a have numerical values.—Put x for

$m + \frac{n}{2}$. Then $x^2 = m^2 + mn + \frac{n^2}{4}$, so that $x^2 = a^2 + \frac{n^2}{4}$ or

$x = \sqrt{a^2 + \frac{n^2}{4}}$. When x has been calculated, m is given by

$m = x - \frac{n}{2}$. Or, without mention of x , we can give m by

$$m = \sqrt{a^2 + \frac{n^2}{4}} - \frac{n}{2}.$$

Use the formula just given to calculate m when $a = 43.28$ and $n = 19.62$.

$$\log a = 1.6363$$

$$\log a^2 = 3.2726$$

$$a^2 = 1873$$

$$\frac{n}{2} = 9.81$$

$$\log \frac{n}{2} = 0.9917$$

$$\log \frac{n^2}{4} = 1.9834$$

$$\frac{n^2}{4} = 96.25$$

$$a^2 + \frac{n^2}{4} = 1969$$

$$\log \left(a^2 + \frac{n^2}{4} \right) = 3.2942$$

$$\log \sqrt{a^2 + \frac{n^2}{4}} = 1.6471 \quad \sqrt{a^2 + \frac{n^2}{4}} = 44.37$$

$$m = 44.37 - 9.81 = 34.56$$

13. By the help of the theorem of Pythagoras show how from two lines a and $\frac{n}{2}$ units long to draw another

$\sqrt{a^2 + \frac{n^2}{4}}$ units long. Take $a = 43.28$ and $n = 19.62$

and use 2 millimetres as unit.—We draw at right angles $PQ = a$, and $PR = \frac{n}{2}$ (Fig. 14). Then $QR^2 = a^2 + \frac{n^2}{4}$ so that QR is the line required.

Show how, knowing a and n , to draw a line of length $m = \sqrt{a^2 + \frac{n^2}{4}} - \frac{n}{2}$.—In Fig. 14, with centre R and radius RP , draw a circle cutting RQ in S . QS is the length m .

Does this suggest a means of finding m from given values of a and n without any calculation at all?—Draw a circle of diameter n units. Draw a tangent PQ of length a units (Fig. 14), and draw through Q the diameter QST . Then $QP^2 = QS \cdot QT$ or $a^2 = QS(QS + n)$, so that QS is the length m that satisfies the equation $a^2 = m(m + n)$.

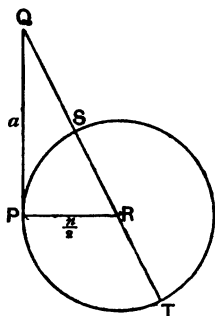


FIG. 14.

14. To sum up, we have seen that a right-angled triangle is determined by any two of the five lengths a , b , p , m , n ; and we have seen how in each case the triangle can be actually made.

Quadratic Equations.

15. We have seen that $m^2 + 3 \cdot 2 m = 102$ if $m = 8 \cdot 6$, or in other words $m^2 + 3 \cdot 2 m - 102 = 0$ if $m - 8 \cdot 6 = 0$. Now subtract m times $m - 8 \cdot 6$ from $m^2 + 3 \cdot 2 m - 102$.—There remains $11 \cdot 8 m - 102$. Now subtract $11 \cdot 8$ times $m - 8 \cdot 6$ from $11 \cdot 8 m - 102$.—There remains $-0 \cdot 5$, or, to the degree of accuracy we are working to, there remains nothing. Thus $(m + 11 \cdot 8) \times (m - 8 \cdot 6) = m^2 + 3 \cdot 2 m - 102$.

Calculate the values of $y = m^2 + 3 \cdot 2 m - 102$ for $m = 0, 4, 8, 12, 16, 20$, and draw the graph of y from $m = 0$ to $m = 20$.—The values are

$m =$	0,	4,	8,	12,	16,	20,
$y =$	-102,	-73,	-12,	80,	205,	362.

Calculate also the values of $z = (m + 11.8)(m - 8.6)$ for the same values of m .—For $m = 12, 16, 20$, the values are $z = 81, 206, 363$, the differences from the corresponding values of y being about what we should expect from the roughness of our approximations. For the other values of m one of the factors is negative, thus when $m = 0$, $z = (11.8) \times (-8.6)$.

We saw some time ago (chap. VI, art. 16) the meaning of a negative sum of money. To have -5 shillings meant to have a debt of 5 shillings. If a boy has 3 sums of -5 shillings, what is the total sum he has?—He owes 3 sums of 5 shillings, that is, 3×5 shillings; that is, he has -3×5 shillings.

So that in this case $3 \times (-5)$ is $-(3 \times 5)$. Let us assume that $(11.8) \times (-8.6) = -(11.8 \times 8.6)$, and generally that $a \times (-b)$ or $(-a) \times b$ is $-(a \times b)$; let us go on this assumption until it lands us in an absurdity or contradiction. If it never brings us to this pass we shall go for ever on this assumption.

16. With this assumption calculate the values of z for $m = 0, 4, 8$.—They are $11.8 \times (-8.6)$ or -101 , $15.8 \times (-4.6)$ or -73 , $19.8 \times (-0.6)$ or -12 ; the differences from the values of y being again accountable by the roughness of our approximations. So far then the assumption is justified.

Now calculate the values of z or $(m + 11.8)(m - 8.6)$ for some negative values of m , namely, $-4, -8, -12, -16, -20$.—When $m = -4$, $z = 7.8 \times (-12.6) = -98$; when $m = -8$, $z = 3.8 \times (-16.6) = -63$; when $m = -12$, $z = (-0.2) \times (-20.6)$, an expression for which we have no meaning.

Work on the assumption that $(-a) \times (-b) = a \times b$, until you find it leads you astray.—With this assumption, for $m = -12$ we have $z = 4$; for $m = -16$ z is $(-4.2) \times (-24.6)$ or 103 ; for $m = -20$ $z = (-8.2) \times (-28.6) = 235$.

Ex. 3. You know that when a, b, c, d are positive numbers $(a + b)(c + d) = ac + ad + bc + bd$, and that this relation can be illustrated by dividing up a rectangle measuring $a + b$ by $c + d$. Give b and d negative values, see what becomes of this relation on our assumption about the product of negative quantities, and whether it can be illustrated by a rectangle.

17. Now calculate the values of y or $m^2 + 3.2m - 102$ for $m = -4, -8, -12, -16, -20$.—When $m = -4$, y is $(-4) \times (-4) + 3.2 \times (-4) - 102$ or $16 - 12.8 - 102$ or -98.8 ; and so for the others, the result being

$$\begin{array}{cccccc} m & -4, & -8, & -12, & -16, & -20, \\ y & -98.8, & -108.8, & -118.8, & -128.8, & -138.8, \end{array}$$

and agreeing closely with the values of z .

Use these values to continue the graph of y so as to show it from $m = -20$ to $m = 20$. Read off from the graph values of m for which $y = 0$.—They are $m = 8.6$ and $m = -11.8$.

The first is the root we had before of the equation $m^2 + 3.2m - 102 = 0$. The other, the negative value of m , is meaningless for the original problem in which m was the length of a line. The same equation might turn up in some other connexion in which a negative root might have meaning.

Could you have obtained these two roots otherwise than from the graph?—Take the expression $m^2 + 3.2m - 102$ in the form $(m + 11.8)(m - 8.6)$. This will be zero if either factor is zero, that is, if $m = 8.6$ or if $m = -11.8$.

Ex. 4. You have frequently used the graph of x^2 for positive values of x . By the aid of the assumption that $(-a) \times (-b) = ab$ draw the graph of x^2 for negative as well as positive values of x .

In the case of sums of money, by attaching a meaning to negative sums we extended the applicability of a formula, and so attained some economy of writing; with practice this economy becomes an economy of thought also. Here again certain assumptions about negative quantities enable us to treat $m^2 + 3.2m - 102$ and $(m + 11.8)(m - 8.6)$ as equivalent for all values of m . In these and other cases of extension of meaning our ultimate justification is the economy of thought that results.

18. Return again to the original calculation* of the root of the equation $m^2 + 3.2m = 102$ and repeat it, remembering that $(-a) \times (-b) = ab$.—We found that $(m + 1.6)^2 = 102 + 1.6^2 = 105$, and that $105 = (10.2)^2$. But since $(-10.2) \times (-10.2) = 10.2 \times 10.2$, we know that $(-10.2)^2 = 105$. Hence the equation $(m + 1.6)^2 = 105$ is satisfied by making

* Page 262.

either $m+1\cdot6 = 10\cdot2$ or $m+1\cdot6 = -10\cdot2$, which give the two values $m = 8\cdot6$ and $m = -11\cdot8$.

Generalize this for the case $m^2 + nm - a^2 = 0$.—We have $(m+n/2)^2 = a^2 + n^2/4$, and since $a^2 + n^2/4$ is equal to

$$\sqrt{a^2 + n^2/4} \times \sqrt{a^2 + n^2/4},$$

and also to $(-\sqrt{a^2 + n^2/4}) \times (-\sqrt{a^2 + n^2/4})$,

we can satisfy the equation by making $m+n/2$ equal to

either $\sqrt{a^2 + n^2/4}$ or $-\sqrt{a^2 + n^2/4}$,

that is, by making m equal to either

$$\sqrt{a^2 + n^2/4} - n/2 \quad \text{or} \quad -\sqrt{a^2 + n^2/4} - n/2.$$

19. Solve the equations :

$$x^2 - 3\cdot2x - 102 = 0,$$

$$x^2 + 3\cdot2x + 102 = 0,$$

$$x^2 - 3\cdot2x + 102 = 0,$$

$$x^2 + 3\cdot2x + 10 = 0,$$

$$x^2 + 3\cdot2x - 10 = 0,$$

$$x^2 - 3\cdot2x + 1 = 0,$$

$$x^2 - 3\cdot2x - 1 = 0,$$

giving in each case as many values of x as you can ; and in any case when you can find no value of x , explain where the method breaks down. In any case where you can find no root, draw the graph of the expression, and see if the graph also fails to give a root.

What is the distinguishing feature of the equations among the above seven that have no root ?—When the last term of the equation is positive and greater than $1\cdot6^2$, there is no root.

Find generally when the equation $x^2 + ax + b = 0$ has roots.—We find that $(x+a/2)^2 = a^2/4 - b$. If $a^2/4 - b$ is positive so that we can find its square root, we shall get roots of the equation. But if b is greater than $a^2/4$, $a^2/4 - b$ is negative and has no square root, and there is no root of the equation.

Calculation of right-angled triangle from various data.

20. We have seen that a right-angled triangle is fixed in shape and size by any two of the lengths a, b, p, m, n of Fig. 11 on p. 260. Incidentally we have seen how m may be calculated from given values of n and a . Show how m having been calculated, b and p may also be calculated.—They are given by $b^2 = n(n+m)$ and $p^2 = mn$. Calculate them for $a = 10.1$ and $n = 3.2$, and compare with the result of drawing and measuring.

21. In Fig. 11 how many right-angled triangles are there? Write down the relation between the three sides of each in terms of a, b, p, m, n . Through how many points can a circle be drawn? From the four points A, B, C, D , choose three in all possible ways to draw a circle through, and in each case write down the relation among a, b, p, m, n that the rectangle property of a circle gives.—The six relations are

- i $(m+n)^2 = a^2 + b^2$;
- ii $a^2 = p^2 + m^2$;
- iii $b^2 = p^2 + n^2$;
- iv $p^2 = mn$;
- v $a^2 = m(m+n)$;
- vi $b^2 = n(m+n)$.

The triangle being fixed by any two of these lengths, it may be possible to calculate the remaining three in terms of those two. We have seen how to calculate m, p, b in terms of n and a . See if, by means of the six relations above, any other pair can be made to serve instead of n and a .

22. The calculation of m, b, p from n and a was made from the equations v, vi, iv. The lengths being determined in this way, the equations i, ii, iii must also be true. Can you prove that any quantities a, b, p, m, n that satisfy iv, v, vi, will also satisfy i, ii, iii?—The equations v and vi give $a^2 + b^2 = m(m+n) + n(m+n) = (m+n)^2$; iv and v give $a^2 = m^2 + mn = m^2 + p^2$; iv and vi give $b^2 = n^2 + mn = n^2 + p^2$.

Can the truth of any three of the six equations be deduced from the truth of the other three?—It can in some cases,

but not in all. For instance, suppose equations ii, iv, v true. These do not contain b , and so, clearly, cannot give us an equation that does contain b . Are the equations ii, iv, v independent, or can one of them be deduced from the others?—We have seen that ii follows from iv and v, and in fact any one of the three follows from the other two.

Verify that any three independent equations from the six (that is, three such that one of them does not follow from the other two) are sufficient to establish the whole six.

23. Two lengths being enough to fix the triangle in shape and size, how many data do we need to fix it in shape alone? Will one length do?—A length will not do, but the ratio of two lengths will. How many ratios do the five quantities a, b, p, m, n give.—We may choose any one of the five for denominator, and each denominator has the other four quantities for possible numerators, so that there are twenty ratios.

Ex. 5. Draw four right-angled triangles whose shapes are fixed by

$$\frac{p}{a} = 0.5; \quad \frac{p}{n} = 3; \quad \frac{m}{n} = 2.3; \quad \frac{a}{n} = 1.$$

Express the other ratios in terms of $\frac{p}{a}$.—By dividing every term in each of the equations i-vi by a^2 we have equations that give $\frac{b}{a}, \frac{m}{a}, \frac{n}{a}$. The ratio $\frac{a}{m}$ is the reciprocal of $\frac{m}{a}$, and we get $\frac{p}{m}, \frac{b}{m}, \frac{n}{m}$ by multiplying $\frac{p}{a}, \frac{b}{a}, \frac{n}{a}$ by $\frac{a}{m}$. And so for any other denominator.

Ex. 6. Give all the other ratios when $\frac{p}{a} = 0.5$.

Ex. 7. A right-angled triangle ABC is made from an equilateral triangle ABD by drawing $BC \perp$ to AD . Calculate the ratios $\frac{AC}{AB}, \frac{BC}{AB}, \frac{BC}{AC}, \frac{BA}{AC}, \frac{AC}{BC}, \frac{AB}{BC}$. What is the size of each angle of the triangle ABC ?

Ex. 8. A right-angled triangle ABC is made by cutting a square $ACBD$ along a diagonal. Calculate the angles of the triangle, and the six ratios of its sides.

24. Can you express the ratio $\frac{p}{a}$ in terms of an angle?—

It is called $\sin B$ (chap. IX, art. 32).

Can you express

any other ratios in terms of B ? $\frac{m}{a}$ is $\cos B$, and $\frac{p}{m}$ is $\tan B$ (chap. XI, arts. 10, 13).

The triangle is fixed in shape by the one ratio $\frac{p}{a}$ or $\sin B$.

So $\cos B$ and $\tan B$ are also fixed. Can you express them in terms of $\sin B$?—The relation $p^2 + m^2 = a^2$ gives

$$\left(\frac{p}{a}\right)^2 + \left(\frac{m}{a}\right)^2 = 1, \text{ or } (\sin B)^2 + (\cos B)^2 = 1,$$

which expresses $\cos B$ in terms of $\sin B$. It gives also

$$1 + \left(\frac{m}{p}\right)^2 = \left(\frac{a}{p}\right)^2, \text{ or } 1 + \frac{1}{(\tan B)^2} = \frac{1}{(\sin B)^2},$$

$$\text{or } (\tan B)^2 = \frac{(\sin B)^2}{1 - (\sin B)^2}.$$

For shortness of writing it is usual to write $\sin^2 B$ instead of $\sin B \times \sin B$, or $(\sin B)^2$, and so with the other ratios; so that the relations given are written:—

$$\cos^2 B = 1 - \sin^2 B;$$

$$\tan^2 B = \frac{\sin^2 B}{1 - \sin^2 B}.$$

Ex. 9. Express $\sin B$ and $\tan B$ in terms of $\cos B$, and $\sin B$ and $\cos B$ in terms of $\tan B$.

Ex. 10. Look up the tables for $\sin 40^\circ$, and use the expressions just given to calculate $\cos 40^\circ$ and $\tan 40^\circ$, and compare with the values given by the tables. Further check all three ratios by drawing a triangle with angles of 90° and 40° , measuring its sides, and calculating the ratios.

25. Suppose you are asked to make a right-angled triangle with a perimeter of 30 cm., that is, with the sum of its three sides 30 cm. in length. How will you do it, and how far is

its form at your disposal?—Draw two lines CX and CY at right angles. Make a loop of thread measuring 30 cm. round. Drive in a pin at C , and another at some point A of CX . Put the loop over these two pins, and stretch the loop to the furthest point on CY that it will reach. Many triangles are possible, for A may be taken on CX anywhere less than 15 cm. from C .

State the condition that the perimeter is 30 cm. in terms of a, b, m, n, p .—It is $a + b + m + n = 30$.

Suppose the triangle is to have a perimeter of 30 cm. and an area of 20 sq. cm.—By means of the loop of thread we can find the length of CB for any position of A , and calculate the area. We draw a graph giving the area as a function of the length CA , and read off a value of CA that makes the area 20 sq. cm.

Another way of finding the triangle of given area and perimeter is to take any value of CA , to find the corresponding value of CB that gives the triangle the required area, and to measure the perimeter of the resulting triangle. We thus get a graph of the perimeter as a function of CA , and from the graph we read off the value of CA that gives the required perimeter.

State in terms of a, b, m, n, p the conditions that the perimeter is to be 30 cm. and the area 20 sq. cm.—They are

$$a + b + m + n = 30 \quad \text{and} \quad ab/2 = 20.$$

26. How would you, without any geometrical construction, calculate a, b, m, n, p for the triangle that has $a + b + m + n = 30$ and $ab/2 = 20$? It will be well by means of the relations that always hold among a, b, m, n, p , to get rid of some of these quantities. Try getting rid of m and n .—We know that $(m + n)^2 = a^2 + b^2$, and the perimeter condition $a + b + m + n = 30$ gives

$$(m + n)^2 = (30 - a - b)^2,$$

so that $(30 - a - b)^2 = a^2 + b^2$.

Can you now, without further regard for m, n, p , find from the relations $ab = 40$ and $(30 - a - b)^2 = a^2 + b^2$ what values a and b must have?—We could take any value for a , calculate from $ab = 40$ the corresponding value of b , and repeat for a number of values of a . We could then calculate the value of $(30 - a - b)^2 - a^2 - b^2$ for each pair of values of a and

b , and show this quantity also in a graph as a function of a . A value of a that makes this expression $(30-a-b)^2 - a^2 - b^2$ equal to 0 gives a triangle for which $a+b+m+n=30$ and $ab/2=20$.

You could also take any value of a , and calculate from $(30-a-b)^2 = a^2 + b^2$ the corresponding value of b . You could then calculate $ab-40$ for each pair of values, and draw the graph of $ab-40$ as a function of a . A value of a that makes $ab-40=0$ gives the triangle we seek.

Having in either way found a , how do you calculate the remaining quantities b, m, n, p ?—The equation $ab-40=0$ gives b . Then $a+b+m+n=30$ gives $m+n$. Then $p=ab/(m+n)$, $m^2=a^2-p^2$, $n^2=b^2-p^2$ give p, m, n .

27. From the equation $(30-a-b)^2 - a^2 - b^2 = 0$ calculate b when $a=2, 4, 6, 8, 10, \dots$ —When $a=2$ the equation is $(28-b)^2 - 4 - b^2 = 0$. We multiply out $(28-b)^2$ and get $784 - 56b + b^2$, so that the equation is $780 - 56b = 0$; and b is $= 780/56$, that is, $= 13.9$. When $a=4$, the equation is $(26-b)^2 - 16 - b^2 = 0$; it simplifies to $660 = 52b$ and gives $b=12.7$. Similarly for other values of a . Can you, without giving a a numerical value, simplify the equation $(30-a-b)^2 - a^2 - b^2 = 0$, so that the one simplification will be available for all numerical values of a ? $(30-a-b)^2$ multiplied out is $900 - 60a - 60b + a^2 + b^2 + 2ab$, so that the equation is $900 - 60a - 60b + 2ab = 0$. This is equivalent to $60b - 2ab = 900 - 60a$, and to

$$b = \frac{900 - 60a}{60 - 2a}.$$

Using the relation in this last form we can substitute any value of a and at once easily calculate the corresponding value of b .

Can you, from the equations $a+b+m+n=30$ and $ab=40$, calculate a and b without the help of graphs?—We have seen that these equations can be put in the form

$$b = \frac{900 - 60a}{60 - 2a} \quad \text{and} \quad b = \frac{40}{a}.$$

Each of these equations gives b when we know a ; and when a has the right value the two equations give the same

value of b . Let us find what value of a will make the two equations give the same value of b . Let us find a value of a that makes

$$\frac{900-60a}{60-2a} = \frac{40}{a};$$

or that makes $(900-60a)a = (60-2a)40$; or that makes $900a-60a^2 = 2400-80a$; or $60a^2-980a+2400 = 0$; or $a^2-\frac{98}{6}a+40 = 0$. We now have the quadratic equation in its simplest form, and we proceed to solve it in the usual way.

$$\begin{aligned} a^2 - \frac{98}{6}a + \left(\frac{49}{3}\right)^2 &= \left(\frac{49}{3}\right)^2 - 40 \\ &= 66.69 - 40 \\ &= 26.69 \end{aligned}$$

$$(a - 8.167)^2 = (5.166)^2$$

so that $a - 8.167 = +5.166$ or $= -5.166$

and $a = 8.167 + 5.166 = 13.333,$

or $= 8.167 - 5.166 = 3.001.$

Take 13.33 and 3 as approximate values for a , show that for each of these values the two equations for b give the same value for b , and find what this value is.

Tabulate all the solutions you have found for a right angled Δ that has a perimeter of 80 cm. and an area of 20 sq. cm.

28. Of the following data for making six right-angled triangles, determine in each case whether the triangle can be made, and whether the number of solutions is definite. When there are an indefinite number of solutions, explain why the two data are not enough to fix the triangle; when the number of solutions is definite, find the number of solutions; and when the number is 0, explain what difficulty occurs in the attempt to find a triangle that satisfies the conditions:

- | | |
|--------------------|------------------------|
| 1. $a+b+m+n = 30,$ | $a+b = 17.$ |
| 2. $ab = 40,$ | $900-60(a+b)+2ab = 0.$ |
| 3. $a+b+m+n = 30,$ | $900-60(a+b)+2ab = 0.$ |
| 4. $a+b+m+n = 30,$ | $ab = 100.$ |
| 5. $a+b+m+n = 30,$ | $a = b.$ |
| 6. $ab = 40,$ | $a = b.$ |

We conclude that any two independent conditions fix a right-angled triangle, that each condition may give the value of one of the quantities a , b , m , n , p , or may give a relation among these quantities.

EXERCISES.

11. It is required to mark out with tape on a plot of grass a rectangle of 26 yards by 12 yards. What ought the length of its diagonal to be to the nearest foot?

Explain how a knowledge of this length would help you to lay out the rectangle truly, and state the geometrical proposition of which you have made use.

12. Draw a triangle ABC , having $\angle A$ a right angle, $\angle B$ 35° , and AB 12 centimetres long. Bisect BC at D and join AD . Measure the length of AC , and calculate the lengths of all the other lines and the sizes of all the other angles in the figure.

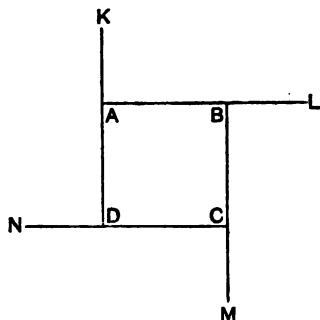


FIG. 15.

13. The frame represented in the rough sketch in Fig. 15 supports a table; $ABCD$ is a square 12 inches in the side, and each prolongation AK , BL , CM , DN of a side is 3 inches; the legs are at K , L , M , N , and the corners of the table are at the same points. What is the shape of the table, and what are its dimensions to the nearest tenth

of an inch? Justify your statement.

14. With your protractor make an angle of 38° , and find by drawing the sine and cosine of the angle.

Check your results by calculating from them $\sin^2 38^\circ + \cos^2 38^\circ$.

15. Figs. 16 and 17 are two different arrangements of five pieces of cardboard (four congruent triangles and a square). Show that Fig. 17 has the form of two squares placed side by side.

Hence or otherwise show how two squares may be joined together, and the resulting figure cut up and re-joined into a single square. What kind of triangle do the sides of the three squares form? Justify your answer.

16. Draw two circles in contact and a common tangent (not the one at the point of contact of the two circles). Draw three more figures, varying the radii. On each figure measure in centi-

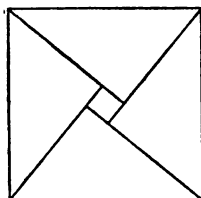


FIG. 16.

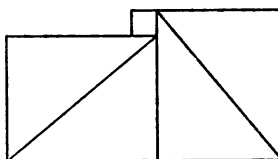


FIG. 17.

metres the radii R and r of the touching circles, and the common tangent T . Make a table like the following, filling the first three columns from your measurements and calculating the last two columns from them.

—	R	r	T	Area of square on T	Area of rectangle $4Rr$
Fig. 1.					
Fig. 2.					
Fig. 3.					
Fig. 4.					

What inference can you draw from the last two columns of your table? Attempt to justify it geometrically.

17. Two ships A and B were steaming at 12 and 18 knots respectively, A going due east and B due north. B crossed A 's direction east of A , and ten minutes afterwards A crossed the track of B , south of B . If D represent the distance in nautical miles between the ships at m minutes after the first event (m being less than 10) prove that

$$13D^2 = 36 + (13m - 4)^2.$$

For what value of m is $(13m - 4)^2$ zero? Is it ever negative? For what value of m is $36 + (13m - 4)^2$ least, and what is this value? What is then the length of D ?

Make a diagram to a scale of 2 inches to a nautical mile, showing the positions of the ships at the moment B crossed A 's direction and at intervals of 1 minute thereafter up to 10 minutes. Measure the distances and plot them on squared paper, representing

1 minute by 1 inch and 1 nautical mile by 2 inches. From this find by inspection the time at which the ships were nearest to one another, and their distance apart at that time, and compare with your previous results.

A knot is a speed of one nautical mile per hour.

18. Make a Δ having angles of 50° and 90° . From it measure the sine, cosine, and tangent of 50° .

Draw any right-angled triangle ABC right-angled at C , and express the sine, cosine, and tangent of B in terms of the sides a , b , c . From known relations between a , b , c express $\sin B$ and $\cos B$ in terms of $\tan B$. In the case of $B = 50^\circ$ verify these relations from the values found above.

19. Make an angle whose tangent is 0.462. Calculate the sine and cosine, and check your results by measurement of your figure.

20. A litre measure is 14.2 cm. high. What must be the height (to the nearest millimetre) of a half-litre measure of the same shape? Among measures of the same shape the volume varies as the cube of the height.

21. A projectile whose weight is w pounds and diameter d inches strikes a wrought-iron plate when moving at the rate of v feet per second. Assuming that the penetration p (in inches) is given by the formula

$$p = \frac{v}{608.3} \sqrt{\frac{w}{d}} - 0.14d,$$

find the penetration when $d = 13.5$, $w = 1250$, and $v = 2016$.

22. A water main is 2 feet in diameter. How many gallons are contained in 1 mile of it? And at how many miles per hour must the water flow in it to supply a population of 63,000 with 38 gallons per head in 12 hours? (1 cubic foot = 6.2 gallons. The area of a circle is 0.79 times the square of the diameter.)

23. Along a line AXB of indefinite length set off $AX = 8$ cm., $XB = 4$ cm., and describe a circle with AB as diameter, naming the centre O . Draw CXD at right angles to AB , cutting the circle at C and D . Calculate the lengths of OC and OX , and show that $XC = \sqrt{32}$ cm. Use your figure to find to the nearest tenth the square root of 32.

If the length of AX be denoted by a and of XB by b , state in its simplest form the value of XC , justifying your statement by geometrical reasoning.

24. The volume of a sphere varies as the cube of its radius. If the mass of Saturn is $1/3502$ that of the Sun, and its density 0.52

that of the Sun, find the diameter of Saturn (supposed spherical), that of the Sun being 866,400 miles.

25. The law for the growth of an electric current in a circuit when the voltage rises suddenly to a value V is given by

$$C = \frac{V}{R} \{1 - e^{-Rt/L}\}$$

where C is the current in amperes, t the time in seconds, and V, R, L are certain constants. Take $V = 100$, $R = 5$, $L = 0.05$, and draw a curve showing the connexion between C and t for values of t ranging from 0 to 0.05. Take $e = 2.718$.

26. The safe width of a dam at a depth of x feet below the water level is $\sqrt{\frac{0.05 \times x^3}{9 + (0.03 \times x)}}$ feet. How wide should a dam be at a depth of 25 feet below the water level?

CHAPTER XVI

CALCULATION OF A TRIANGLE

1. We have seen (chap. V) that a triangle is determined by any three of the six quantities a, b, c, A, B, C ; except the three angles. We have seen that the sum of the three angles is always 180° , so that a knowledge of two angles is as good as a knowledge of three; the three amount only to two independent conditions. So that we may say that any three independent quantities of the six a, b, c, A, B, C , fix the triangle.

We have seen (chap. XV) for a right-angled triangle that any set of data that enable us to draw the triangle enable us also to calculate the triangle. Is this true of all triangles?

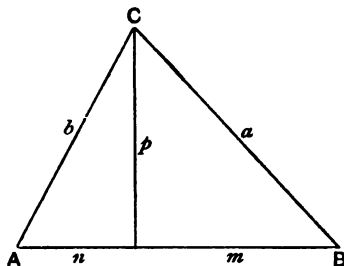


FIG. 1.

2. Let us suppose we know a side a and two angles A and B of the triangle ABC (Fig. 1). Draw the $\perp p$. Suppose the lengths of the various lines of the figures to be given

in centimetres or any other unit by the numbers a, b, m, n, p , as shown in the figure. Write down all the relations you know among a, b, p, m, n, A, B , and see if you can by means of them calculate the parts b, c, C, m, n, p , from the known parts a, A, B .—First, we know

$C = 180 - A - B$. Then $\cos B = \frac{m}{a}$ and $\sin B = \frac{p}{a}$ give

m and p . Knowing p we get b from $\frac{p}{b} = \sin A$ and n from $\frac{n}{b} = \cos A$. Finally, $c = m + n$ gives c .

Ex. 1. Carry out these calculations for the case $a = 14.72$, $A = 58.4$, $B = 46.3$, the length being in centimetres and the angles in degrees. Check your results by drawing and measuring.

3. In general we do not want m , n , p , but only b , c , C . Can you rearrange the equations found so as to give b and c without mention of m , n , p ?—These equations give

$$b = \frac{p}{\sin A} \quad \text{and} \quad p = a \sin B, \quad \text{so that} \quad b = \frac{a \sin B}{\sin A}. \quad \text{For } c$$

we have $c = m + n$, $m = a \cos B$, $n = b \cos A$, so that $c = a \cos B + b \cos A$. If you wanted c without calculating b , can you express c in terms of a , A , B ?—Putting

$$\frac{a \sin B}{\sin A} \quad \text{for } b \text{ we have } c = a \cos B + \frac{a \sin B \cos A}{\sin B}$$

$$\text{or } c = a \times \frac{\sin A \cos B + \cos A \sin B}{\sin A}.$$

This expression for c is heavy to use in calculation. Can you find a better? Try the relations that would result from dropping a $\perp$ from A or B instead of from C .—The $\perp$ let fall from B is equal to $a \sin C$, and also to $c \sin A$, so

that $c = \frac{a \sin C}{\sin A}$, C being known at once from $C = 180 -$

$A - B$. The $\perp$ let fall from A gives $c = \frac{b \sin C}{\sin B}$, not so good because b is not among the original data.

Use these relations, which may be written in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

to find b and c when $a = 14.72$, $A = 58.4$, $B = 46.3$.—Looking out $\sin 58.4$ and $\sin 46.3$ in the tables we get

$$b = \frac{14.72 \times 0.7230}{0.8517},$$

which we may simply multiply and divide out, or we may use logarithms, thus:

$$\log 14.72 = 1.1679$$

$$\log 0.7230 = 0.8591 - 1$$

$$\hline 1.0270$$

$$\log 0.8517 = 0.9303 - 1$$

$$\log b = 1.0967, \therefore b = 12.49.$$

And so for c .

In this work you have looked up two entries for each angle, namely for B

$$\sin 46.3 = 0.7230$$

$$\log 0.7230 = 0.8591 - 1.$$

But in another place the tables give the logarithm of $\sin 46.3$ as a single entry. This is the table called, not very appropriately, 'logarithmic sines.' The actual entry is

$$\log \sin 46.3 = 9.8591,$$

which is greater by 10 than the value already found. To enter $0.8591 - 1$ would make the table clumsy; for compactness the logarithm of every sine, cosine, and tangent is shown in the table with 10 added.

4. To survey a piece of land two points A and B 500

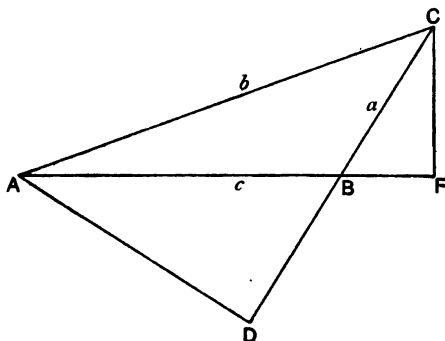


FIG. 2.

metres apart are taken; and to fix a point C the angles $CAB = 19^{\circ}.5$ and $CBA = 123^{\circ}$ are measured (Fig. 2). Cal-

culate the distances of C from A and B .—Since $A + B + C = 180^\circ$, $C = 37^\circ.5$. The relation $a/\sin A = c/\sin C$ gives (in metres) $a = 500 \sin 19^\circ.5 / \sin 37^\circ.5 = 274$. The angle B is obtuse, and we have no meaning for the sine of an obtuse angle; but if we draw $AD \perp$ to CB produced, we have $AD = c \sin ABD$ and also $= b \sin C$. The angle $ABD = 180^\circ - B = 57^\circ$, so that $b = 500 \sin 57^\circ / \sin 37^\circ.5 = 689$.

You had for Fig. 1 the relation $c = a \cos B + b \cos A$. What corresponding relation is there for Fig. 2?—Draw $CF \perp$ to AB produced. Then $b \cos A$ is AF . We have no meaning for $\cos B$, since B is obtuse, but $a \cos CBF$ is BF . So that $c = b \cos A - a \cos CBF$.

5. The meaning of sine and cosine can be extended to cover obtuse angles. Let OX and OY be two lines at right angles (Fig. 3). Let us agree to consider all lengths measured along OX or parallel to OX positive, and all lengths measured along or parallel to the opposite direction OX' negative. In the same way all lengths in the direction OY are to be considered positive and all in the reverse direction OY' negative. Let a line OP lie at first along OX and then turn in the direction of OY through an angle of $19^\circ.5$. Draw $PM \perp$ to OX . The sine of $19^\circ.5$ is MP/OP , and the cosine of $19^\circ.5$ is OM/OP , OM and MP being positive because they lie in the directions OX and OY . Use this to find $\sin 19^\circ.5$ and $\cos 19^\circ.5$, and compare with the values given in the tables.

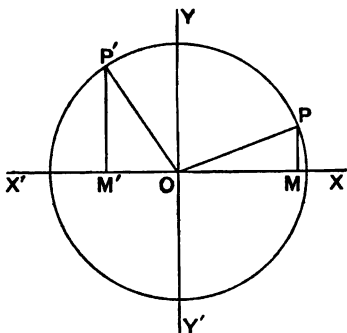


FIG. 3.

So far our definitions agree with the former ones. Now let OP turn from OX through 128° to OP' , and draw $P'M' \perp$ to $X'OX$. $M'P'$ is still positive, but OM' is negative as it lies in the negative direction OX' . Draw the figure

making $OP' = 10$ cm., and measure OM' and $M'P'$. Still taking as our definitions

$$\sin 123^\circ = M'P'/OP', \quad \cos 123^\circ = OM'/OP',$$

give their values.— $P'M'$ is 8.4 cm. and OM' is 5.4 cm., so that $\sin 123^\circ = 8.4/10 = 0.84$ and $\cos 123^\circ = -5.4/10 = -0.54$.

How will you find $\cos 123^\circ$ from the tables?—The tables give angles from 0° to 90° . The angle $P'OM'$ is $180^\circ - 123^\circ = 57^\circ$. The cosine of an acute angle is positive, and $\cos 57^\circ$ is simply the length OM' divided by the length OP' , without regard to the fact that OM' is negative. $\cos 57^\circ$ is $5.4/10$ or 0.54 , by measurement; or more accurately from the tables, $\cos 57^\circ = 0.5446$. Thus $\cos 123^\circ = -0.5446$.

6. In view of these extended definitions, see what corresponds for an obtuse-angled triangle (Fig. 2) to the relations $a/\sin A = b/\sin B = c/\sin C$ and $c = a \cos B + b \cos A$ for an acute-angled triangle.— $\sin B$ is CF/CB , and is positive, so that $a \sin B = CF = b \sin A$. Also AD is $= c \sin B$ and is also $= b \sin C$, so that the relation $a/\sin A = b/\sin B = c/\sin C$ is true for this triangle also. $\cos B$, however, is negative, and is $= -BF/a$, so that $BF = -a \cos B$. Thus $c = AF - BF = b \cos A + a \cos B$, the same relation as before.

Our extended definitions thus prove convenient. It saves space in the memory, or in the notebook, to be able to use the old formulas instead of keeping a second set.

Ex. 2. Draw a few triangles and measure their sides and angles. a, b, c being the lengths of the sides of a triangle in centimetres, and A, B, C the values of the angles in degrees, calculate and compare $\frac{a}{A}, \frac{b}{B}, \frac{c}{C}$ for each triangle.

Calculate the angles and sides of one of the right-angled triangles made by cutting in half an equilateral triangle 10 cm. in the side, and compare $\frac{a}{A}, \frac{b}{B}, \frac{c}{C}$. Do the same for the triangle made by cutting a square 10 cm. in the side along a diagonal.

Are the sides of a triangle proportional to the opposite angles?

7. In the case of triangles fixed by two sides and an angle, what classes are to be distinguished (chap. V)? Given a, b, A , how will you calculate B, C, c ?—The equation

$a/\sin A = b/\sin B$ gives B . Then $A + B + C = 180^\circ$ gives C , and $c/\sin C = a/\sin A$ gives c .

Draw the triangle for the four cases $a = 12.48, 6.76, 3.42, 2.84$, all four cases having $A = 20$ and $b = 10$. Then calculate the triangle for each case.—Remembering that every value of $\sin B$ that gives an acute angle for the value of B gives also an obtuse angle, we have the following results:

a in cm.	B in degrees	C in degrees
12.48	$\left\{ \begin{array}{l} 15.9 \\ 164.1 \end{array} \right.$	$\left\{ \begin{array}{l} 144.1 \\ \text{impossible} \end{array} \right.$
6.76	$\left\{ \begin{array}{l} 30.4 \\ 149.6 \end{array} \right.$	$\left\{ \begin{array}{l} 129.6 \\ 10.4 \end{array} \right.$
3.42	90.0	70.0
2.84	impossible . .	impossible

In the case when $B = 164.1$, the sum of A and B is greater than 180, so that C is impossible and the only triangle possible for $a = 12.48$ has $B = 15.9$ and $C = 144.1$. In the case when $a = 2.84$ the value of $\sin B$ is greater than 1, and no angle has a sine greater than 1, so that there is no triangle with $a = 2.84$. These results agree with the results of drawing.

Of all possible values of a , which give two triangles, which give one, and which none, when $b = 10$ and $A = 20$? Generalize the result?—When a is greater than or equal to 10 there is one triangle; when a lies between 10 and 3.42 there are two triangles; when $a = 3.42$ one triangle; when a is less than 3.42 no triangle. The general results are:—

$a > b$	1 triangle
$b > a > b \sin A$	2 triangles
$a = b \sin A$	1 triangle
$a < b \sin A$	0 triangle

8. Given b, c, A , how will you calculate a, B, C ? Write down again all the relations between $a, b, p, \dots$, and choose the useful ones.— $\sin A = p/b$ gives p , and $\cos A = n/b$ gives n . Then $m = c - n$ gives m . Then $\tan B = p/m$ gives B . And $m/a = \cos B$, or $p/a = \sin B$, gives a .

Solve the triangle $A = 20$, $b = 14$, $c = 6$.—We have $p = 14 \sin 20 = 4.8$; $n = 14 \cos 20 = 13.2$. This gives n greater than c , a case we did not contemplate.

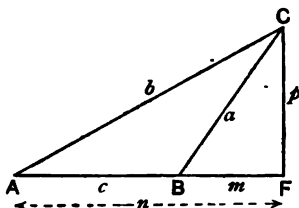


FIG. 4.

Draw a triangle (Fig. 4) in which n is greater than c , and see how you would solve it.—Denote the exterior angle at B by Q . p and n are given as before. m is now $n - c$. Q is given by $\tan Q = p/m$, and B is $180 - Q$. a is given by $m/a = \cos Q$ or $m/a = -\cos B$ or $p/a = \sin Q$

or $p/a = \sin B$. For the case in question this gives $m = 7.2$, $\tan Q = 4.8/7.2 = 0.67$, $Q = 34$, $a = 7.2/\cos 34 = 8.7$.

What meaning would you suggest for $\tan B$, when B is obtuse?—In accordance with the extended definitions given to the sine and cosine (art. 5), let us suppose OP to rotate from OX through the angle B and drop $PM \perp$ to OX (Fig. 3); then $\tan B$ is MP/OM ; the directions OX and OY being positive, the opposite directions negative. When the angle is obtuse MP is positive and OM negative, so that the tangent is negative.

How will this definition apply to the solution of the triangle $A = 20$, $b = 14$, $c = 6$?— $\tan B = -p/m = -4.8/7.2 = -0.67$, and by drawing an angle whose tangent is -0.67 we find the angle to be 146° . Or the table gives $0.67 = \tan 34$, so that $B = 180 - 34 = 146$. We can thus solve the triangle without mention of Q .

Write down the formulas you would use to solve this triangle, without mention of Q , and compare them with those you gave for an acute-angled triangle.—The formulas differ only in the sign of m .

And if you reckon m positive when F lies on the same side of B as A , and negative when it lies on the other side?—The formulas are then identical in form.

Here, again, by our extended definition of $\tan B$ we have the formulas for the solution of a triangle exactly the same for acute- and obtuse-angled triangles.

9. There remains the case in which the three sides a , b , c are given, and the angles A , B , C are to be calculated. What

expressions have you for A and B in terms of a, b, p, m, n (Fig. 1)?— $\cos A = n/b$, $\sin A = p/b$, $\cos B = m/a$, $\sin B = p/a$.

So that if you can calculate m, n, p , you know how then to calculate A and B . Write down all the relations between a, b, p, m, n, c , and see if you can find m, n, p from a, b, c .—The relations are $a^2 = m^2 + p^2$, $b^2 = n^2 + p^2$, $c = m + n$. Let us try to get rid of m and p , and so have an equation for n . The second and third equations give $p^2 = b^2 - n^2$ and $m = c - n$. We put these values for p and m in the first equation and have $a^2 = (c - n)^2 + (b^2 - n^2)$ or $a^2 = c^2 - 2cn + b^2$ or $n = (b^2 + c^2 - a^2)/2c$. Knowing n we can then calculate m, p, A, B, C .

The following way is rather more convenient. Since $a^2 = p^2 + m^2$ and $b^2 = p^2 + n^2$, it follows that $a^2 - b^2 = m^2 - n^2$. Since $m + n = c$, $m^2 - n^2 = (m - n)(m + n) = c(m - n)$. So we calculate m and n from $m - n = (a^2 - b^2)/c$ and $m + n = c$; then A and B from $\cos A = n/b$ and $\cos B = m/a$; and C from the angle sum property. Use this for the case $a = 5.846$, $b = 8.722$, $c = 9.104$.—

$$n - m = \frac{(b + a)(b - a)}{c} = \frac{14.57 \times 2.876}{9.104} = 4.603,$$

$$n + m = c = 9.104,$$

so that $n = 6.853$ and $m = 2.250$.

Then $\cos A = n/b = 6.853/8.722$ and $A = 38.2$,

$$\cos B = m/a = 2.250/5.846 \text{ and } B = 67.4.$$

Finally $C = 180 - A - B = 74.4$.

Check these results by drawing.

10. If you wanted to know C only and not A and B , how would you shorten the work?—We should not draw the $\perp$ from C but from A (or from B). The former procedure would then bring us first to values of $\cos C$ and $\cos B$ (or of $\cos C$ and $\cos A$).

Find the greatest angle of the triangle when the sides are 6.840, 8.722, 13.764.—The angle from which the $\perp$ is drawn is the last one we come to, so we draw the $\perp$ from one of the smaller angles (Fig. 5). The triangle is obtuse, so our formulas may need revision. n is negative when F and B lie on opposite sides of A , so that if the numerical length of

AF is k , $n = -k$. We have now $a^2 - b^2 = m^2 - k^2$ and $m = c + k$, so that $a^2 - b^2 = (m + k)c$ or $m + k = (a + b)(a - b)/c$.

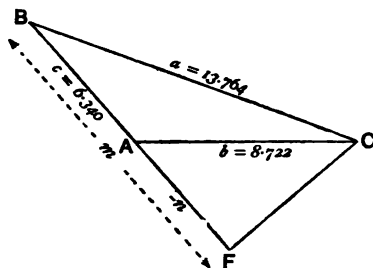


FIG. 5.

If we use n instead of k , the equations for m and n are $c = m + n$ and $m - n = (a + b)(a - b)/c$, as before. Thus,

$$m - n = (a + b)(a - b)/c = 22.486 \times 5.042/6.340 = 17.87,$$

$$m + n = c = 6.340,$$

whence $m = 12.10$ and $n = -5.76$;

and $-\cos A = -n/b = 5.76/8.722 = \cos 48.7$.

Thus A is an obtuse angle that has the same numerical value of the cosine as 48.7 , so $A = 180 - 48.7 = 131.3$.

11. Can you tell from the given values of a , b , c whether a triangle is right-, acute-, or obtuse-angled, without calculating the angles?—We know already that if $a^2 + c^2 = b^2$, the angle B is right. Also we have seen that in all cases $a^2 - b^2 = (m - n)c$ and $c = m + n$. Hence, in all cases, $a^2 - b^2 = (2m - c)c$ or $2mc = c^2 + a^2 - b^2$. Now F may lie (see Fig. 6) to the left of B , in which case m is positive; it may lie to the right of B , in which case m is negative; and it may coincide with B , in which case m is zero. When $c^2 + a^2 > b^2$ we know from $2mc = c^2 + a^2 - b^2$ that m is positive, that F lies to the left of B , and therefore the angle B is acute. When $c^2 + a^2 < b^2$ we know that m is negative, F lies to the right of B , and $\angle B$ is obtuse. When $c^2 + a^2 = b^2$ the formula shows that $m = 0$, that F coincides with B , and $\angle B$ is right; which we knew already.

Need you in this way consider each angle separately to

determine the nature of the triangle?—Only the greatest angle can be right or obtuse, and the greatest angle is

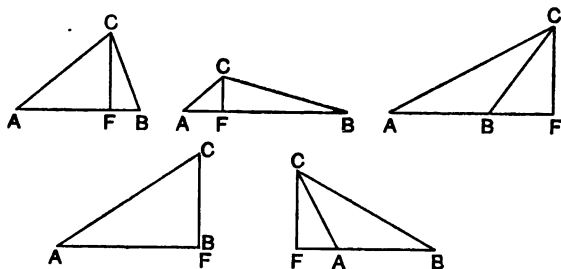


FIG. 6.

opposite to the greatest side. So let us call the greatest side b ; then the triangle is acute-, right-, or obtuse-angled, according as $c^2 + a^2$ is $>$, $=$, or $<$ b^2 .

Ex. 3. Taking as the lengths of two sides of a triangle 5 and 10 cm., and taking in succession as the length of the third side 16, 14, 12, 10, 8, 6, 4, 2 cm., draw all the triangles you can, and find by calculation what kind of triangle each is. When no triangle can be drawn, explain why.

Ex. 4. Two rods 5 and 10 cm. long are hinged together, and the free ends joined by an elastic thread. Show in a graph the length of the thread as a function of the angle between the rods.

Give another graph showing the area of the square whose side is equal to the thread, as a function of the angle between the rods. Show on the same diagram the sum of the squares on the two rods. For what values of the angle between the rods is the single square greater than the sum of the two, and for what values is it less?

12. In Fig. 6 BF is called the projection of BC on BA . In general, if $\perp s$ Pp and Qq are let fall from the two ends P and Q of a line on another line X (Fig. 7), pq is the projection of PQ on X . Also positive directions being chosen for PQ and X , for instance, those marked by arrow-heads in the figure, the projection pq is considered positive if it runs in the positive direction of X and negative if in the other. Can you express the projection in terms of the angle between PQ and X ?—If A is the angle between the positive directions and l the length of PQ , then $pq = l \cos A$,

and the algebraic sign looks after itself, since $\cos A$ is positive when A is acute and negative when A is obtuse.

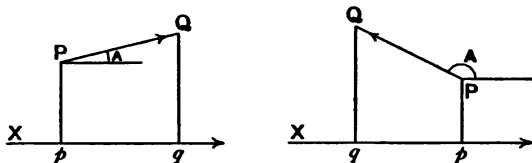


FIG. 7.

Express the projection n of b on c in terms of the angle A , and in place of the relation found between a, b, c, n give one between a, b, c, A .—Since $n = b \cos A$, the relation $2nc = b^2 + c^2 - a^2$ becomes $2bc \cos A = b^2 + c^2 - a^2$. Write down an expression connecting a, b, c, B , and one connecting a, b, c, C .

13. Draw any triangle and the three $\perp$ s from the angles on the opposite sides. Find, by experiment, whether the three $\perp$ s are usually concurrent, that is, meet in one point.

In any triangle ABC draw the $\perp$ s BE and CF (Fig. 8) meeting in O . Pick out any set of four points through

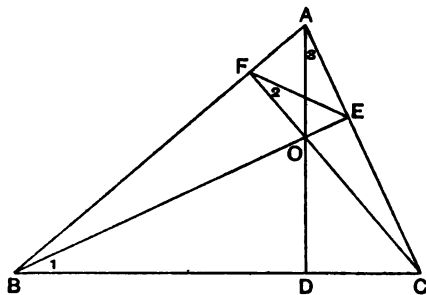


FIG. 8.

which a circle will pass; a number of points through which a circle will pass are called concyclic.—The angles BFC and BEC are both right, so that the circle on BC as diameter passes through B, F, E, C . For the same reason the circle on AO as diameter passes through A, F, O, E . Join

AO and FE , and mark all equal angles that you can discover. Produce AO to meet BC in D . What can you tell of the angle ADB ?—We know that the angles marked 1 and 2 are equal because B, F, C, E are concyclic; and that the angles 2 and 3 are equal because A, F, O, E are concyclic. The equality of the angles 1 and 3 follows, and shows D, B, A, E to be concyclic. Therefore $\angle ADB$ is $= \angle AEB$ and so is a right angle. Thus AO produced is the $\perp$ from A on BC ; that is, the three $\perp$ s are concurrent or meet in a point.

From measurements of your figure calculate the areas $OA.OD$, $OB.OE$, $OC.OF$, $AF.AB$, $BF.BA$, $BD.BC$, $CD.CB$, $CE.CA$, $AE.AC$. What relations hold (experi-

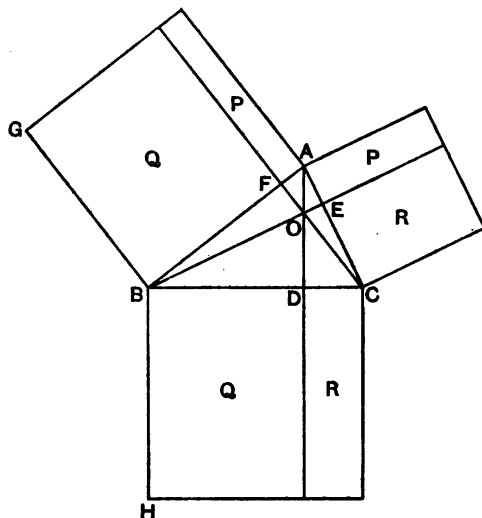


FIG. 9.

mentally) among them? Consider AF as the projection of AC ; OE as the projection of OC or OA , and so on; and so justify the experimental results.

14. Draw any triangle ABC with the squares on the three sides (Fig. 9). Draw the $\perp$ from each vertex on the opposite

side and produce it to divide the square on that side into two rectangles. What rectangle property do you get from the circle through B , F , E , C ? — That $AF \cdot AB = AE \cdot AC$.

Pick out a rectangle drawn on your figure equal to each of these. Repeat for the two circles on AB and AC as diameters. What area of the figure is equal to $a^2 + b^2 - c^2$? — We have seen that the two rectangles marked P are equal, as are the two marked Q , and the two marked R . We have then to subtract $P + Q$ which makes c^2 from the sum of $Q + R$ (which is a^2) and $P + R$ (which is b^2). The remainder is $2R$.

Express R in terms of sides and angles of the triangle. — One rectangle marked R is $a \times CD$, the product of a and the projection of b on a ; the other is $b \times CE$, the product of b and the projection of a on b . The projection CD is $= b \cos C$ and the projection CE is $= a \cos C$. So from either rectangle we have $R = ab \cos C$.

You thus arrive again at the equation $c^2 = a^2 + b^2 - 2ab \cos C$; which is an extension of the theorem of Pythagoras that $c^2 = a^2 + b^2$ when $C = 90^\circ$. In Fig. 9 join GC and HA , and compare the triangles GBC and ABH with one another and with the rectangles marked Q , and so give a proof after the manner of Euclid's proof of the theorem of Pythagoras, that the square on one side of an acute-angled triangle is less than the sum of the squares on the other two sides by twice the rectangle contained by one of these sides and the projection on it of the other.

15. Reconsider the preceding article for an obtuse-angled triangle.

Calculation of the area of a triangle.

16. A set of data that fix a triangle, fix the area and enable us to calculate it. The most useful expressions for the area are in terms of the three sides and in terms of two sides and the included angle. Express the area in terms of b , c , A . — The area is half the product of the base and the height, $pc/2$ (Fig. 1 or 4). And $p = b \sin A$, so that the area is $\frac{1}{2} bc \sin A$.

Find an expression in terms of the three sides. — We must express p in terms of the sides. We have $p^2 = a^2 - m^2$

while m is given by $2cm = c^2 + a^2 - b^2$. This expression for m shows that $(a+m)2c$ is equal to

$2ac + c^2 + a^2 - b^2$ or $(c+a)^2 - b^2$ or $(c+a+b)(c+a-b)$;
and that $(a-m)2c$ is equal to

$2ac - c^2 - a^2 + b^2$ or $b^2 - (c-a)^2$ or $(b-c+a)(b+c-a)$.

Therefore $4c^2p^2$ or $2c(a+m) \cdot 2c(a-m)$ is =

$$(c+a+b)(c+a-b)(b+c-a)(b+c-a).$$

And the area or $\frac{1}{2}cp$ is

$$\sqrt{\frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{c+a-b}{2} \cdot \frac{a+b-c}{2}}.$$

The symmetry of this expression in a, b, c is a check on its correctness. We get the same result no matter which side we choose as base. Use the formula to calculate the area of a triangle whose sides are 13, 10, 8 cm. Also measure one of the angles and calculate the area from $\frac{1}{2}bc \sin A$. Also draw a perpendicular and use $\frac{1}{2}pc$. See how closely your results agree.

A quadratic equation.

17. The relation $a^2 = b^2 + c^2 - 2bc \cos A$ between a, b, c, A , enables us to calculate any one of these four quantities when we know the other three. Use it in the following cases, and compare it for rapidity with methods already given:

$$(1) a = 4.36, \quad b = 9.00, \quad c = 6.32;$$

$$(2) b = 7.20, \quad c = 5.68, \quad A = 118°;$$

$$(3) a = 3.64, \quad b = 7.42, \quad A = 28°.$$

Consider how, when a, b, A are given, the different cases arise of two solutions, one solution, and no solution. The equation for c is $c^2 - 2bc \cos A = a^2 - b^2$ or

$$(c - b \cos A)^2 = a^2 - b^2 \sin^2 A.$$

Since $\sin^2 A + \cos^2 A = 1$, the right hand side of the equation may be written $a^2 - b^2 \sin^2 A$; and the equation itself

$$(c - b \cos A)^2 = a^2 - b^2 \sin^2 A.$$

For this to give any value at all for c , the right side must be positive, that is, a must be $> b \sin A$.* For such values of a the equation always gives two values of c . But

* Discuss the case $a = b \sin A$.

a negative value of c is useless as the side of a triangle. Both values of c will be positive when

$$b \cos A > \sqrt{a^2 - b^2 \sin^2 A},$$

that is, when $b^2 \cos^2 A > a^2 - b^2 \sin^2 A$

or $b^2 (\cos^2 A + \sin^2 A) > a^2$

or $b^2 > a^2$

or $b > a$.

And when $a > b$ there is only one positive value of c and one triangle.

Triangle given by co-ordinates.

18. The position of each vertex of a triangle may be given by its distances from two $\perp$ lines or axes OX and OY (Fig. 10).

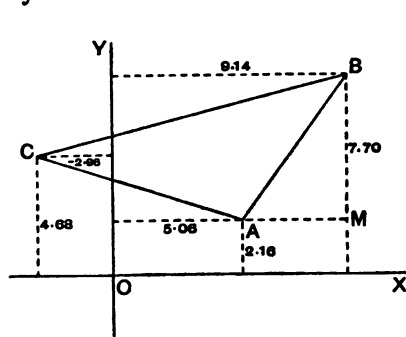


FIG. 10.

These distances are called co-ordinates. In Figure 10 the co-ordinates of A are 5.06 and 2.16, the co-ordinate measured $\parallel$ to OX being given first; the co-ordinates of B are 9.14 and 7.70; the co-ordinates of C are -2.96 and 4.68. The co-ordinate -2.96 is negative because it is not measured in the direction OX , but in

the opposite direction. Similarly, any co-ordinate measured in the opposite direction to OY is negative.

Can you calculate the length AB and the angle that AB makes with OX (or with a parallel to OX)?—If AM is drawn $\parallel$ to OX we have a right-angled triangle in which

$AM = 9.14 - 5.06 = 4.08$ and $BM = 7.70 - 2.16 = 5.54$.

Whence $AB^2 = 4.08^2 + 5.54^2 = 47.8$, and $AB = 6.88$. And $\tan BAM = 5.54/4.08$, so that $\angle BAM$ is $58^\circ 6'$.

Calculate the lengths of the other sides and the angles they make with OX , and so the angles of the triangle. Check by drawing.

In this case a triangle is fixed by six data, the six co-ordinates. In the cases previously discussed three data fixed a triangle. What makes the difference?—The six co-ordinates fix the triangle in position as well as form. Three conditions are enough to fix the form by itself.

19. In most of the previous discussions the triangle has been given by the specification of three of the elements a, b, c, A, B, C . But the three conditions may be given in various ways. Discuss, in each of the eight following cases, whether a triangle can be made to satisfy the conditions, and whether there are an indefinite number of different triangles that satisfy the conditions or only a finite number. When there are an indefinite number of solutions, show that we have not three independent conditions.

1. $a + b + c = 40$, $A + B + C = 180$, $A = B$.
2. $a + b = 20$, $b + c = 30$, $A + B + C = 170$.
3. Triangle to be equilateral and have a perimeter of 40 cm.
4. Triangle to be isosceles, have base of 10 cm., and height of 18 cm.
5. Base = 14 cm., height 10 cm., vertical angle 70° .
6. $b + c = 9$, $c + a = 11$, $a + b = 10$.
7. $a + b = 12$, $a = c$, $a \sin B - b \sin A = 0$.
8. $B + C = 110$, $C + A = 130$, $A + B = 120$.

EXERCISES.

5. To find the breadth of a straight river two points A and B are taken on one bank 115 yards apart. From these a point C on the further bank is observed, and it is found that $\angle ABC = 40^\circ$ and $\angle BAC = 64^\circ$. Draw a figure on a scale of 1 inch to 20 yards, and find from it the breadth of the river.

6. A point P lies 3 miles from a point O in a direction 81° north of east, another point Q lies $5\frac{1}{2}$ miles from O in a direction 57° north of east. Calculate the distance between P and Q to the nearest tenth of a mile.

7. Prove the accuracy of the following geometrical construction for determining the roots of the quadratic $x^2 - px + q = 0$: 'Draw

$OX = p$; draw XZ and OY perpendicular to OX , making $XZ = q$ and $OY = \text{unity}$; join YZ , and on it describe a semicircle, cutting OX in A and B . OA and OB are the required roots.'

Solve the equation $x^2 - 3.5x + 2 = 0$ in this way.

8. For printing a circular, a printer's estimate is 1s. 9d. for 50 copies, or 2s. 8d. for 100. Presuming each of these estimates to consist of (i) a charge for setting up the type, independent of the number of copies printed, (ii) a charge for printing and paper, proportional to the number of copies printed; find what his estimate for printing 1000 copies would be.

9. The given numbers are known to follow approximately the law $y = a + bx$. Determine the probable values of a and b .

$y = 0.65, 0.91, 1.26, 1.69, 1.77, 2.13.$

$x = 1.5, 3.0, 5.0, 7.5, 8, 10.$

10. In a certain experiment the following values of x and y were obtained:—

$x \dots$	1.0	1.7	2.5	3.8	4.2	6	7.8	10
$y \dots$	4.30	5.38	6.63	8.64	9.26	12.05	14.84	18.25

Determine the probable values of y when x was 3, 5 and 8, and exhibit them in a table.

Assuming that there is a simple law connecting x and y , find the law, and check your results by means of it.

11. The sides of a rectangle are p and q inches in length. From a point within the rectangle four straight lines are drawn perpendicular to the sides; find the sum of these lines in terms of p and q . Prove that the sum is the same for all positions of the point within the rectangle.

If the point is outside the rectangle, and perpendiculars are dropped from it on the sides, produced if necessary, show that the sum of the perpendiculars can still be expressed in terms of p and q by regarding some of the lines as positive and others as negative.

12. Multiply together $x+a$, $x+b$, $x+c$ and arrange your result by bracketing together like powers of x .

Describe your result in words and deduce from it the result of multiplying $x+a$, $x-b$ and $x+c$ together.

13. Criticize the following reasoning :

$25 - 45 = 16 - 36$, that is, $5^2 - 2 \times 5 \times \frac{3}{2} = 4^2 - 2 \times 4 \times \frac{3}{2}$;
therefore
 $5^2 - 2 \times 5 \times \frac{3}{2} + (\frac{3}{2})^2 = 4^2 - 2 \times 4 \times \frac{3}{2} + (\frac{3}{2})^2$, that is, $(5 - \frac{3}{2})^2 = (4 - \frac{3}{2})^2$;
therefore $5 - \frac{3}{2} = 4 - \frac{3}{2}$;
therefore $5 = 4$.

14. Show that

$$x^2 + px + q = \left(x + \frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2}\right) \left(x + \frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}\right)$$

and give the values of x for which $x^2 + px + q = 0$.

What values of x satisfy the equation when

$$p = -0.1 \text{ and } q = -0.3 ?$$

15. The tangent of an angle which lies between 90° and 180° is -2 : state and carry out a construction for finding the angle, and determine its sine and cosine.

16. Draw a field $ABCD$ in which $AB = 80$ metres, $AC = 94$ m., $AD = 78$ m., $\angle CAB = 42^\circ$, $\angle CAD = 46^\circ$.

Could you find the area of a quadrilateral field if you knew the lengths of the sides ? Why ?

17. A tunnel has to be bored through a mountain connecting two places A and B . To get from A to B over the mountain I have to walk 4 kilometres upwards at an angle of 23° and then 2 kilometres downwards at an angle of 34° with the horizon, the path being all in one vertical plane. Find the length of the tunnel, the difference of level of its two ends, and the inclination of the tunnel to the horizontal.

18. It is noticed that when the elevation of the Sun is 22° the shadow of the summit P of a mountain falls at Q on the seashore. A distance QR of 800 feet is measured horizontally at right angles to PQ and the angle PRQ is then found to be 73° . Calculate the height of P above sea level.

Also find the height of P by drawing.

19. From the top of a hill of height 874 metres above the level of an adjoining lake the angles of depression of the top of another hill and of its reflection in the lake were found to be 11° and 26° ; (the hill top in the reflection is vertically below the actual hill top and is as far below the lake surface as the actual top is above). Find the height of the second hill from a drawing to a suitable scale.

Also calculate the height of the second hill, having given that its ratio to the height of the first hill is $\sin(d' - d) \div \sin(d' + d)$, where d and d' are the two angles of depression.

20. Solve the equations $y - 0.4x = 3.2$, $x + 0.6y = 12$, and verify

Among measures of the same shape the volume varies as the cube of the height. A litre is 1.76 pints to the nearest hundredth.

28. A is a hill top, B and C two points at sea level 3,000 feet apart. It is found that $\angle ABC = 55^\circ 30'$ and $\angle ACB = 82^\circ$, while the elevation of the hill top viewed from C is 12° . By drawing find the height of the hill above sea level.

29. The area of the opening under the arch of a bridge is sometimes calculated from the formula $\frac{4V}{3} \sqrt{(0.626V)^2 + C^2}$ where V is the height of the arch and C half the distance between the ends of the arch. Calculate this area to two significant figures when $V = 12$ feet and $C = 16$ feet.

The area evidently lies between that of a rectangle of the same base and height and that of a triangle of the same base and height. Use this fact as a rough check on your result.

30. Find out with as little work as possible whether a 2-inch pipe through which water is running at 4 miles an hour is large enough to fill a bath holding 15,000 gallons within 10 hours. A 2-inch pipe has, roughly, a cross-sectional area of $3\frac{1}{4}$ square inches, and a cubic foot is 6.23 gallons.

31. To find the height of a tower on the further side of a river, a base of 165 yds. was measured up the slope of a hill from a station on the same level as the bottom of the tower. At the upper station the angles of depression of the lower station, the top, and the bottom of the tower, were respectively 47° , 19° , and 27° . Determine the height of the tower in any way you please.

32. The corner C of an estate was specified as being 90 links from a certain reference tree, in a direction $40^\circ 15'$ south of east. The corner C was moved later 150 links to the south. What was its correct specification then?

33. A survey line AB is run as shown in the sketch (Fig. 11). The length of each part in links is shown. The angle written against each part means the angle it makes with the north, measured round from the north to the east. Find the direction (or bearing) and distance of B from A .

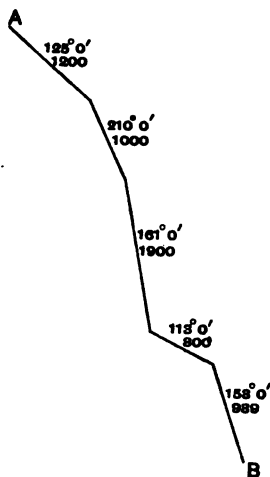


FIG. 11.

34. Draw any triangle. Taking any side as base, make on it an isosceles triangle equal in area to the original triangle. Measure the two perimeters, and see which is greater, and establish the result by general reasoning.

Take now the isosceles triangle, consider one of the equal sides as base, and make on this base an isosceles triangle equal in area to the former isosceles triangle. Is the perimeter increased or diminished?

Show that by continuing in this way the perimeter of the triangle is in general continually reduced, the area remaining unchanged. Is there any case in which the perimeter cannot be reduced?

What is the shape of a triangle of minimum perimeter for given area?

35. Take a loop of string and stretch it with three pins into a triangle. Try by moving a pin to increase the area of the triangle. Then move another pin, and so on. Establish by general reasoning how each pin must be moved to increase the area. In what case is it impossible to increase the area?

What is the shape of a triangle of maximum area for a given perimeter?

CHAPTER XVII

SOLID GEOMETRY

1. We found experimentally (chap. VII, art. 14) that a line standing on a plane might be $\perp$ to the plane, that is, $\perp$ to every line in the plane that passed through the foot of the line. Must this remain simply the result of experiment, or is it capable of proof?

Draw a pretty large square $ABCD$, and on each side of it as base draw an isosceles triangle, all four triangles being congruent (Fig. 1). Cut out the whole figure and fold it

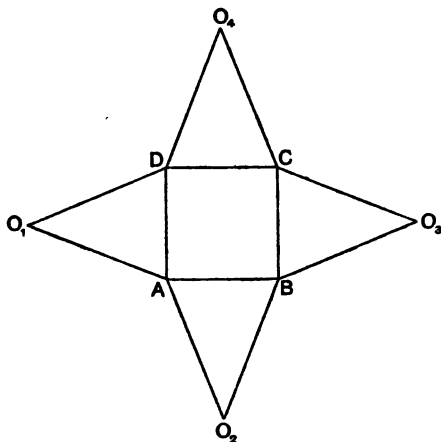


FIG. 1.

along the sides of the square. Turn the triangles about the creases till their vertices meet and you have the pyramid $OABCD$ (Fig. 2). Does the height of the isosceles triangles

matter?—The vertices of the triangles will not come together if the height is too little; the height must be greater than half the side of the square.

2. Now draw the diagonals AC and BD of the square base of the pyramid, meeting in M . How does the line OM lie with regard to the base $ABCD$?—It appears to

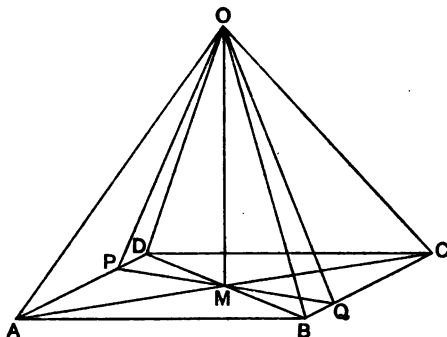


FIG. 2.

be $\perp$ to the base.

Can you prove it $\perp$ to any lines on the base?— OM will be shown $\perp$ to AMC if we show $\Delta s OMA$ and OMC congruent. OA is the line in which O_1A and O_2A of Fig. 1 unite, OC is the line in which O_1C and O_2C unite, and we know all the lines $O_1A, O_2A, O_3C, \dots$ are equal; so that $OA = OC$. Again, the diagonals of a square bisect one another, so that $MA = MC$. Lastly, MO is a side of each triangle. Therefore the triangles OMA and OMC are congruent, and the angles OMA and OMC are equal, and OM is $\perp$ to AMC . A similar proof shows OM to be $\perp$ to DMB .

Draw across the base, through M , any line PQ . Can you show this line to be $\perp$ to OM ? Show $\Delta s CMQ$ and AMP congruent and $CQ = AP$, then show $\Delta s COQ$ and AOP congruent and $OQ = OP$; and then proceed by the comparison of $\Delta s OMP$ and OMQ to show that OM is $\perp$ to PMQ .

It has been proved that OM is $\perp$ to every line through M in the plane $ABCD$; so that the existence of a perpendicular to a plane is established.

3. How would you cut out a sheet of paper to fold into a pyramid on a base that is rectangular but not square?—The isosceles triangles of Fig. 1 must have the sides O_1A , O_2A , O_3B ,... all equal; but the Δ s are not now all congruent, the bases being unequal. How does the line from the vertex of the pyramid to the centre of the rectangular base lie?—It is $\perp$ to the base, and the proof runs as before.

If you know a line to be $\perp$ to one line in a plane do you know it to be $\perp$ to the plane?—No. For instance, the axle of a cart is a line in a horizontal plane, and every spoke is $\perp$ to the axis, but the spoke is not usually $\perp$ to the horizontal plane. In fact each spoke is $\perp$ to the horizontal plane only twice in a revolution of the wheel, once pointing right upwards, and once pointing right downwards.

If you know a line to be $\perp$ to two lines in a plane, do you know it to be $\perp$ to the plane, that is, $\perp$ to every other line in the plane?—Suppose $OM \perp$ to AC and DB (Fig. 3). Cut off MA , MB , MC , MD , all of the same length. Then Δ s OMA and OMC have $MA = MC$, $\angle OMA = \angle OMC$, and OM is a side of each, so that the triangles are congruent and $OA = OC$. In the same way OB , OD are all equal, and O is the vertex of a pyramid on the rectangle $ABCD$ as base. Now take in the plane any other line and let it cut DA in P and BC in Q . By considering Δ s MDA and MBC , and then Δ s MDP and MBQ , we find $MP = MQ$; by considering Δ s ODA and OBC , and then Δ s ODP and OBQ , we find $OP = OQ$. Therefore Δ s OMP and OMQ are congruent, and OM is $\perp$ to PMQ . Thus OM is $\perp$ to every line in the plane that contains AC and DB .

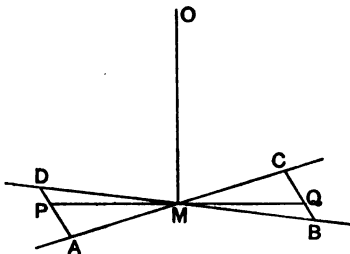


FIG. 3.

4. Suppose that in Fig. 3 a line SMR is drawn $\perp$ to MO . Will this line lie in the plane $ABCD$?—Think of the plane

K that contains MO and SMR . It cuts the plane $ABCD$ in some line TMU . So we have the plane K containing the line MO , the line SMR which is $\perp$ to MO , and the line TMU . Further, MO being $\perp$ to every line in the plane $ABCD$, is $\perp$ to TMU . Since in the plane K there is only one line through $M \perp$ to MO , SMR and TMU must be the same line. That is, any line $SMR \perp$ to MO must lie in the plane $ABCD$.

5. Suppose AMC and OM two rods soldered together at M at right angles. And suppose that the rod AC , while keeping its ends at A and C , turns round and carries OM with it. What kind of surface does MO (produced indefinitely) sweep out?—It looks like a plane.

Can you prove that a line turning about a fixed line and always $\perp$ to it traces out a plane?—Suppose MO and MO' in Fig. 4 two positions of a line turning about AM and always $\perp$ to it. Think of

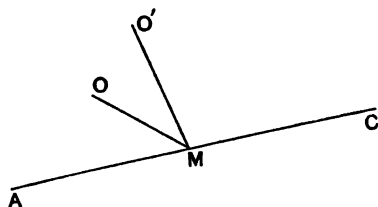


FIG. 4.

the plane P that passes through OM and $O'M$. Since MA is $\perp$ to OM and to $O'M$ it is $\perp$ to every line in P that passes through M ; in other words, AM is $\perp$ to the plane P . And we know that any line $\perp$ at M to MA lies in this plane which is $\perp$

to MA . Therefore any third position $O''M$ of the rotating line lies in the plane P determined by OM and $O'M$. That is, OM sweeps out the plane P . More shortly put, two positions of OM determine the plane $P \perp$ to AM ; and any third position of OM , being $\perp$ to AM , also lies in the plane P . So that OM sweeps out the plane $\perp$ to AM .

6. How does the end O of the rod MO move as MO rotates round AMC ?— O is all the time in the plane P , and is all the time at the same distance from M .

We first met the circle as the path of a point at a constant distance from the foot of a tree (chap. I). It appeared later that for the path to be a circle the point had to remain on the ground (chap. VII, art. 4); the complete locus of a point moving at a constant distance from a fixed point being

a sphere. Give now a complete definition of a circle.—It is the path of a point that moves in a plane and keeps at a constant distance from a fixed point in the plane.

And what is the path of O ?—By the definition the path of O is a circle.

7. Fig. 5 shows a triangular pyramid, in which the edges AB , BC , CD have the same length, say 10 cm.; also the angle BCD is right, and AB is $\perp$ to the plane BCD . Suppose

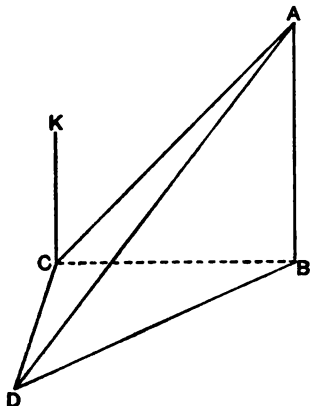


FIG. 5.

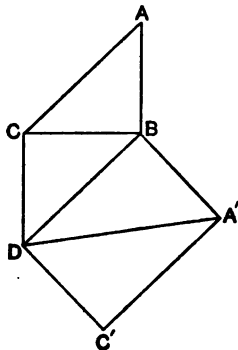


FIG. 6.

the pyramid has a covering of paper, which is slit along BA , AC , CD , and spread out. Draw carefully this spread out covering (Fig. 6), arranging the size of the angle $A'CD$ so that the covering will fold up again and form a pyramid.

Calculate the lengths of the various edges, use these lengths to draw the figure carefully on stiff paper, and make a pyramid.—Since AB is $\perp$ to the plane BCD , $\angle ABC = 90^\circ$, and $AC^2 = AB^2 + BC^2 = 200$ sq. cm., so that $AC = 14.14$ cm. The angle BCD being right, we get in the same way $BD = 14.14$. That AB is $\perp$ to BCD gives also $\angle ABD = 90^\circ$, so that $AD^2 = AB^2 + BD^2 = 800$, and $AD = 17.32$.

Do these calculations tell you anything of $\angle ACD$?—They give us $AD^2 = AC^2 + CD^2$, so that the angle ACD must be right.

8. Let us now take our paper pyramid, set it on BCD as base, and at the point C put in a pin CK parallel to AB . What do parallel lines mean?—When all the lines dealt with were in one plane (chap. II, art. 2) we saw that when two lines were $\parallel$, a $\perp$ to one was always $\perp$ to the other, and the $\perp$ distance between the $\parallel$ lines was the same all along. Later, in dealing with figures not all in one plane (chap. VII, art. 13) we found that for lines to be $\parallel$, they must in addition lie in one plane.

We had $AB \perp$ to the plane BCD , and have drawn $CK \parallel$ to AB . Is CK also $\perp$ to BCD ?—If we can show CK to be $\perp$ to two lines in BCD , we shall have shown it to be $\perp$ to the plane. Since CB falls across the parallels BA and CK , the angle KCB is $180^\circ - \angle ABC$ or 90° . We have still to show CK to be $\perp$ to CD . CK being $\parallel$ to AB is in the plane ABC . Now DC was made $\perp$ to BC , and we have seen that it is also $\perp$ to CA ; it is therefore $\perp$ to the plane ABC that contains these lines, and therefore to CK which lies in this plane. Thus CK has been shown $\perp$ to two lines in the plane BCD and so $\perp$ to the plane. So we have proved that if one line is perpendicular to a plane all lines parallel to it are also perpendicular to the plane.

9. State and prove a converse to this theorem.—All perpendiculars to a plane are parallel to one another. Take our paper pyramid again, and now suppose the pin CK set $\perp$ to the plane BCD (Fig. 5). CK and CD are therefore at right angles. We know also that CD is $\perp$ to the plane ABC . Therefore CK lies in this plane. We now have CK and BA in the same plane and both at right angles to BC ; they are therefore parallel.

Show how, given any two $\perp$ s to a plane, to arrange a pyramid about them so as to prove them parallel.

10. The top of your desk is a plane. Set up $\perp$ to it a pencil or penholder by means of a pin in the end of it. Taking the cover of a book for a second plane, try in what variety of ways you can place a plane through the top of the pencil. Then set up a second equal pencil $\perp$ to the desk. In what variety of ways can you place a plane through the tops of the two pencils?—The plane may rotate round the line joining the two tops of the pencils, and any position it

rotates to will do.

If a third equal pencil is added, what variety is left?—None; the plane must rotate till it contains the third pencil top; this position is the only possible one. The only motion possible is one of sliding over the pencil tops.

11. Call the pencils AB , CD , EF (Fig. 7). Compare the triangles ACE and BDF .— AB and CD being $\perp$ to the same plane, the desk, are $\parallel$ to one another. And they are equal. Therefore $ABDC$ is a parallelogram, and AC and BD are also equal and parallel. In the same way AE and BF are equal and parallel, and so are CE and DF ; so that the triangles are congruent.

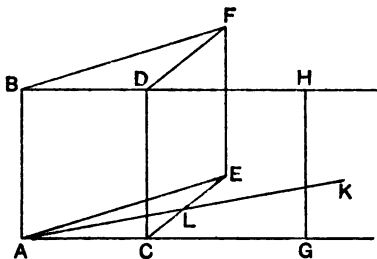


FIG. 7.

How do the pencils stand with regard to the second plane, the cover of the book, which we will call Q ?— $ABDC$ is a rectangle, so that AB is $\perp$ to BD ; in the same way AB is shown to be $\perp$ to BF . It is $\perp$ to two lines in the plane Q , and so $\perp$ to Q . In the same way the other pencils are shown to be $\perp$ to Q ; or otherwise, we know them to be $\parallel$ to AB , and they are therefore $\perp$ to Q .

Let us call the plane of the desk P . Can you tell anything of the perpendicular distance apart of P and Q elsewhere than at the pencils?—At any point G in AC suppose GH drawn $\parallel$ to AB and meeting Q in H . GH therefore lies in the plane $ABCD$ and BD produced will pass through H . Now $ABHG$ is a parallelogram since $AB \parallel GH$ and $AG \parallel BH$. So that $GH = AB$, and GH being $\parallel$ to AB is $\perp$ to the two planes P and Q . So that GH is the distance apart of the planes, and the distance apart anywhere on a prolongation of AC , CE , or EA is the length of a pencil.

Find the distance apart measured from any point K in the plane P .—Join K to A , cutting CE in L . Then we know that the distance apart at L is the same as at A , and we find the distance at K from those at L and A just as we

found that at G from those at C and A . So that the distance apart is everywhere the same.

Such planes, whose perpendicular distance apart is everywhere the same, are said to be parallel planes.

12. Just as the points B, D, F determined a plane, so do B, A, C , or nearly any three points. When are three points not enough to determine a plane?—A plane is not determined by passing through A, C, G , for it may turn about the line ACG . Generally three points in a line do not fix a plane. Name some other sets of points in Fig. 7 that do fix planes.

How would you measure the angle between two planes, say between ABC and ABE ?—On the analogy with an angle contained by lines we might suppose the plane ABC to turn about the pencil AB till it lies flat on ABE , and call the angle turned through the angle between the planes.

How would you measure this angle with a protractor? Your proposal needs to be made more definite. —The turning brings AC to lie along AE ; we could take the angle CAE turned through by the line AC as the measure of the angle between the two planes.

Is the result the same as if you took the angle turned through by BD ?—Yes, for $\triangle ACE$ and BDF are congruent, so that $\angle DBF = \angle CAE$.

Would it do to take the angle between the two positions of BC ?—By experiment that would be a smaller angle than CAE . If $AC = AE$, this angle made by BC would be CBE , and we can show it to be smaller than CAE . For CAE is the vertical angle of an isosceles triangle on base CE and with sides of length CA . The $\triangle CBE$ has the same base, but longer sides of length CB .

13. Then give a measure of the angle between two planes that is the same for all ways of taking the measurement. —Suppose any third plane drawn at right angles to the intersection of the given planes. The angle between the lines in which the third plane cuts them is the angle between the given planes. This measure is always the same; for let us call AB the intersection of the given planes, and call C and E two points on their intersections by the third plane. Draw CD and EF parallel and equal to AB .

Then as before we find that BDF is a plane $\perp$ to AB , the intersection of the given planes, and that the angle DBF is equal to $\angle CAE$. Thus we get the same measure of the angle between the two planes whether we take the third plane through A or through B .

Position of electric lamp.

14. An electric lamp L hanging from the centre C of a ceiling by a wire w is drawn aside and held so by two strings s and t , that go from the lamp to two corners A and B of the ceiling (Fig. 8). Make a model by attaching three strings to three points at the same level, e.g. three corners of an open box. For different lengths of the two strings, what is the locus of all the points the lamp may occupy?

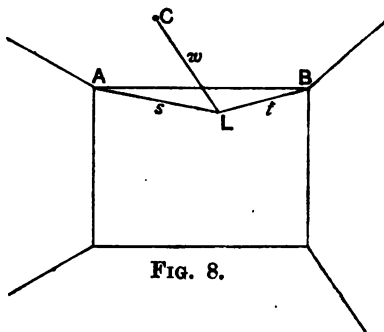


FIG. 8.

—A sphere whose radius is w and whose centre is the point C where the wire is attached to the ceiling.

What is the locus when the lengths s and w are given?—The locus is the points common to the sphere of centre C and radius w and the sphere of centre A and radius s , and it looks like a circle.

Suppose AC joined and LM drawn $\perp$ from L to AC (Fig. 9). The lengths w and s being given,

does the position of M vary for different positions of the lamp?

—The triangle ACL is fixed in size and shape by AC , s , w ; thus $\angle CAL$ is fixed. The $\triangle LAM$ is fixed in size

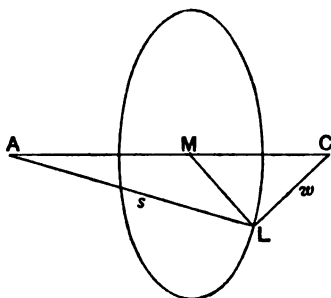


FIG. 9.

and shape by s , $\angle LAM$, $\angle LMA$; thus the length AM is fixed. So that M is at the same distance from A for all positions of L , and ML remains in the same plane $\perp$ to AMC . Since the constancy of $\triangle ALM$ shows the length LM also constant, the locus of L is a circle.

When the length t is given also, what possibilities are there for L ?— L must be at an intersection of the circle just discussed and the sphere with centre B and radius t . It may happen that there are two intersections, or one, or none. When there is no intersection the strings are not of suitable length to support the lamp.

15. When there are two intersections, are they both suitable positions for the lamp? One position is in the room. The other is above the ceiling so that it would not light the room; further the strings would have to be stiff to hold the lamp there, they would have to be rods and not strings.

When you see the image of a lamp in a mirror, how does the image lie with respect to the lamp?—As far as we can judge the image lies as far behind the mirror as the lamp is in front, and the line from lamp to image is $\perp$ to the mirror.

Let us adopt this as a definition of image even when the surface is not a reflecting one. Are the two intersections L_1 and L_2 of the sphere and circle the images one of the other in the ceiling? that is, is the line L_1L_2 perpendicular to the ceiling and bisected by the ceiling?

Take first the question of perpendicularity of the line L_1L_2 (Fig. 10). Can you name two planes of which L_1L_2 is

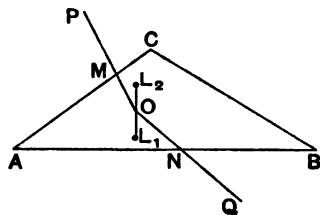


FIG. 10.

the intersection?—We have seen that when the distances w and s are given ML rotates in a plane $\perp$ to AMC ; let us call this plane P . By considering the locus of L when s and t are given but not w , we find the path of L to lie in a plane $\perp$ to AB , say the plane Q . The points L_1 and L_2 lie in each of these planes

P and Q , and therefore lie on the intersection of these planes P and Q ; so that the line L_1L_2 is the intersection.

16. We have seen that the line on which the two possible positions L_1 and L_2 of the lamp lie is the intersection of two planes P and Q , P being $\perp$ to AMC and $Q \perp$ to ANB . What angles do these planes make with the ceiling?—Let OM be the intersection of the plane P with the ceiling (Fig. 11) and MD a line in the plane P and $\perp$ to OM . MC is also $\perp$ to OM , so that the angle between P and the ceiling is $\angle DMC$. And $\angle DMC$ is right since CM is $\perp$ to the plane P ; so that P is $\perp$ to the ceiling. In the same way the plane Q is $\perp$ to the ceiling.

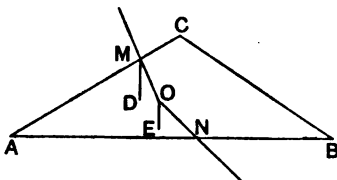


FIG. 11.

Let O be the point where P , Q , and the ceiling all three meet. If OE is a line in the plane P and $\perp$ to OM , how does OE lie with regard to the ceiling?— P is $\perp$ to the ceiling, that is, OE is $\perp$ to a certain line on the ceiling; and OE is also $\perp$ to OM . So OE is $\perp$ to the ceiling. If OF was drawn in the plane Q and $\perp$ to ON , how would it lie with regard to the ceiling?—It would also be $\perp$.

Then are OE and OF the same line?—Suppose $OE \perp$ to the ceiling and OG any other line. Suppose the plane through OE and OG to meet the ceiling in XOY (Fig. 12). Then EOX and EOY are right angles, so that GOX and GOY are not, and GO is not $\perp$ to XOY , and therefore not $\perp$ to the ceiling. Thus there is only one $\perp$ to the ceiling at O , and OE and OF are the same line. This line OE or OF is therefore the intersection of the planes P and Q , that is, it is the line L_1L_2 . And we have seen that it is $\perp$ to the ceiling.

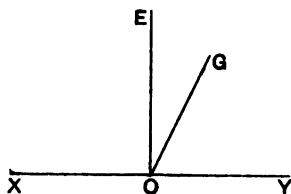


FIG. 12.

17. One of the conditions necessary to make L_1 and L_2 images of one another is established: the line L_1L_2 is $\perp$ to the ceiling. Can you establish the other, namely that L_1L_2

You have now AN and NO . Only OL remains.—The triangle LOA (Fig. 10) has $\angle O$ right, OA has just been calculated, and AL is the given length s ; so that OL is given by $OL^2 = LA^2 - OA^2$.

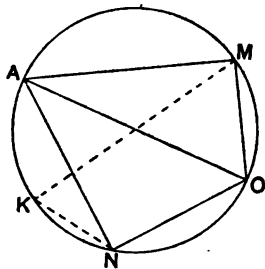


FIG. 15.

19. It is seen that we can calculate all we want about the figure. The methods could perhaps be improved upon. For instance, what do you know of the four points $ANOM$?—They lie in a circle on AO as diameter (Fig. 15).

And if KM is a diameter of this circle, what do you know of $\triangle KMN$?— $KM = OA$ and $\angle MKN = \angle MAN$.

So you can calculate MN from

$$MN^2 = MA^2 + NA^2 - 2MA \cdot NA \cos \angle MAN,$$

and then AO or KM from

$$MN = KM \sin \angle MKN.$$

If you happen to want to find LO only, this is a considerable saving.

Ex. 1. Taking $AB = 17$, $BC = 14$, $CA = 12$, $LA = 13$, $LB = 11$, $LC = 9$, all in centimetres, find by drawing the distance of L from the plane ABC .

20. Assuming that a triangular pyramid, whose height is h cm. and whose base has an area of S sq. cm., has a volume of $\frac{1}{3}hS$ c. cm., show how you could calculate the volume of the pyramid from the lengths of the six edges.—This is the problem of the electric lamp over again. Suppose L the vertex of the pyramid and ABC the base (Fig. 10). We have seen how to calculate LO , the height of the pyramid. And we knew before how to calculate the area of a triangle from the lengths of the sides.

Position of chair, not on floor.

21. Some time ago (chap. II) we discussed the conditions necessary to fix the position of a chair on the floor. How many were needed? Give the conditions in two or three

different ways. If now we remove the restriction that the chair must stand on the floor, how many conditions are necessary? How many conditions fix the position of one point, say one leg, of the chair in space?—We could fix it, as we fix the lamp, by the intersection of three spheres, that is by the distances from three points. Or we could fix it by three co-ordinates, such as the distances from the floor, the west wall, and the south wall. Or in other ways; in every case three conditions.

22. How many conditions will the second leg take?—It will take a total of three, like the first leg. But one condition, its distance from the first leg, is already known; so that it needs only two more.

And for the third leg how many?—The first two legs being placed, the third has already two loci fixed for it; it lies on a sphere of known radius, of which the first leg is centre, and on another sphere with the second leg as centre. One condition more fixes it.

How many more conditions for the fourth leg?—The fourth leg, and every other point of the chair, is already fixed by its distances from the three legs already placed.

Summarize and generalize your results.—The position in space of any solid body is fixed by six conditions. It is fixed as soon as three points are fixed, the first of which takes three conditions, the second two, and the third one. The position of any fourth point may be specified by its distances from these three.

Conditions that fix a framework.

23. Suppose we are dealing with a body whose size and shape have to be fixed as well as its position. For instance, return to the electric lamp, and suppose ABC (Fig. 8) a triangular framework of rods from which the lamp hangs, and suppose we are required to determine all the lengths $AB, \dots AL, \dots$, as well as the position of the whole framework. How many conditions will fix the dimensions of the framework, and how many its position?—For the dimensions six, one for each length; for the position of the framework six, as we found for the chair. And if we

considered dimensions and position together?—The fixing of each point takes three co-ordinates or other conditions, the four points take twelve conditions.

24. Suppose, again, that we have a framework $DABC$ (Fig. 16) made of six rods hinged together; the old figure with the strings supporting the lamp replaced by rods. Make a model with rods. If one rod DC is removed, what degree of freedom has the frame; that is, if we keep one part ABC still, what motion is possible for the rest?— DAB can rotate about AB , and the position of DAB is settled when the distance DC is given; that is, the system has one degree of freedom.

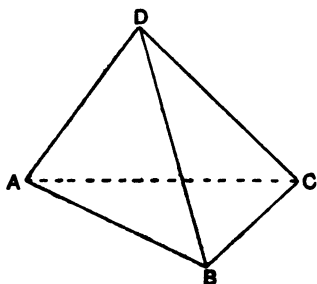


FIG. 16.

When a second rod DB is removed also, what freedom is there?— DA may point in any direction. To give one distance DB removes part of its freedom, to give DC as well settles the position of DA . There are two degrees of freedom.

Give another way in which the position of AD might be specified.—Suppose a plane to coincide with ABC , and then turn about AB till it contains D ; the angle turned through gives one condition. Suppose a line in this plane to turn about A from coincidence with AB to coincidence with AD ; the angle it turns through gives a second condition, and the two specify the position of AD . Or the position AD may be specified by the angles DAB and DAC .

25. If the two rods DC and AB are removed from the complete frame, what freedom has the frame?—The motion is not easy to describe; let us discuss the amount of freedom. Giving one distance DC removes part of the freedom, and giving the second distance AB removes it all. Reckoning as before that the number of degrees of freedom is the number of constraints that must be applied to remove the freedom, we have two as the number of degrees of freedom.

If DC , AB and AD are removed so that AC , CB and BD are left, what freedom is there?— AC being held still, CB may point in any direction; that is, CB has two degrees of freedom. The position of CB being settled, BD may point in any direction, that is, BD has two degrees of freedom; a total of four degrees for the system. Otherwise, the rigid tetrahedron with AC fixed has one degree of freedom, and the removal of the three rods adds three more degrees of freedom.

26. Take a system of five points joined together by rods.

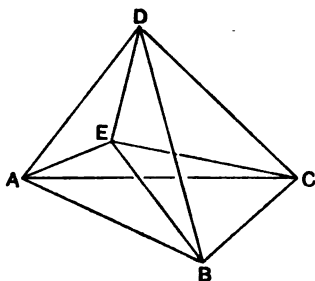


FIG. 17.

How many conditions fix its size and shape?—Begin with the point A (Fig. 17); it may be put anywhere. Next take B ; it may be put anywhere at the proper distance from A , one condition. C must be at the proper distances from A and B , two conditions. D 's position is fixed by the distance from A , B , C , three conditions. E 's position is also fixed by the distances from A , B , C , three conditions.

The total number of conditions is nine.

Check this by counting the conditions to fix the system in size, shape, and position, and the number to fix it in position only.

27. If the five points are joined in every possible way how many rods are there? If you were given ten rods from which to make the frame, could you in general fit the frame together?—The frame is fixed by nine rods. If we follow the same order of building up as before, the distance apart of D and E is settled before we try to add the tenth rod. The tenth rod will fit in only if it happens to have the right length.

Suppose a frame made of five points joined by ten rods. What freedom is given by the removal of one rod?—None, since the remaining nine rods settle the form. And by the removal of a second rod?—This gives one degree of

freedom, and the removal of every further rod adds a degree of freedom up to a certain point. Beyond that point it is not constraints but moving parts that we are removing.

Ex. 2. If n points are joined, two and two, in all possible ways by rods, how many rods are there, and how many are wanted to fix the form of the frame? How many conditions does it take to fix the position of the frame, and how many to fix form and position?

Ex. 3. Remove the rods in any order from a tetrahedral frame (four points joined by six rods) and discuss at what point you begin to remove moving parts and not constraints.

Ex. 4. A doll is jointed with ball and socket joints at the shoulders, elbows, hips, and knees. How many degrees of freedom has it?

Tetrahedron.

26. Consider the triangular pyramid or tetrahedron $OABC$ (Fig. 18). If it has a thin cover which is slit along OA OB and OC , and spread out, what will it look

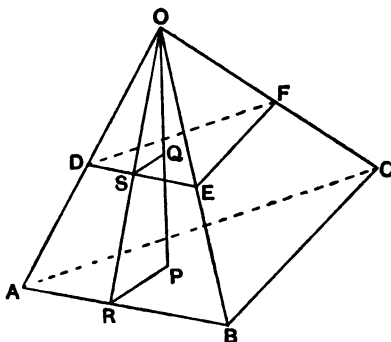


FIG. 18.

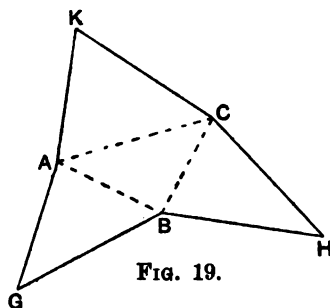


FIG. 19.

like? If such a piece of paper as Fig. 19 shows is cut out and folded along the dotted lines AB , BC , CA , in what circumstances can $G H K$ be brought together? From such a piece of paper make a model of a tetrahedron and use it for the following discussion.

Let a plane parallel to the base ABC cut the pyramid in the triangle DEF . Compare the pyramids $OABC$ and $ODEF$ as far as you can.—Consider the lines AB and DE .

They never meet, however far produced, for they are in parallel planes ABC and DEF . At the same time they are in the same plane OAB ; so they are parallel lines. There-

fore Δs OAB and ODE are similar and $\frac{OD}{OA} = \frac{OE}{OB} = \frac{DE}{AB}$;

similarly each of these ratios is equal to $\frac{OF}{OC}$, $\frac{EF}{BC}$ and $\frac{FD}{CA}$.

Whence also DEF and ABC are similar triangles. Thus the two pyramids are bounded each by four triangles, the triangles being similar in pairs.

29. Is the angle between two faces of one pyramid equal to the corresponding angle of the other pyramid? For shortness denote the faces of $OABC$ by o, a, b, c , each small letter denoting the face opposite the corresponding capital. The faces a, b, c are faces of the small pyramid also; denote its base by p .—The angles between a and b , between b and c , between c and a are the same for the two pyramids. We have to see if $\angle oc = \angle pc$.^{*} Suppose OP drawn $\perp$ from O on the base o . In the plane o draw $PR \perp$ to AB , and consider the plane through O, P, R , which we will call m . The $\perp$ at R to o is $\parallel$ to the $\perp$ PO , and therefore in the same plane m . The plane m is therefore $\perp$ to the intersection AR of the faces c and o , and the angle PRO is the angle between c and o . Further, let m cut p in SQ ; then DSE , being $\parallel$ to AB , is also $\perp$ to m , so that OSQ is the angle cp . And RP and SQ , being in the same plane m , and in $\parallel$ planes o and p , are parallel lines. Therefore the angles OSQ and ORP are equal; and the angle between two faces of one pyramid is equal to the corresponding angle of the other.

30. Suppose the area of the base ABC to be S sq. cm., the height OP to be H cm., and the distance OQ of the section from the vertex to be h cm. What is the area of the section?—Suppose the section has an area of s sq. cm. Then

$$\frac{s}{S} = \left(\frac{DE}{AB}\right)^2 = \left(\frac{DO}{AO}\right)^2 = \left(\frac{h}{H}\right)^2 \quad \text{or} \quad s = \frac{h^2}{H^2} S.$$

Now let OP be divided into n equal parts, and through every point of division let a horizontal plane be drawn, that

^{*} $\angle oc$ means angle between planes o and c .

is, a plane $\parallel$ to the base which we will suppose horizontal. The pyramid is thus cut up into slabs. What is the area of the section by the r^{th} plane, that is, the plane at distance $\frac{rH}{n}$ from the vertex?—The area is $(\frac{rH}{n}/H)^2 S$ or $\frac{r^2}{n^2} S$.

If these slabs had vertical faces in place of their sloping faces, we could calculate their volume. Let us suppose pieces added to their sloping faces so that each slab, instead of narrowing from base to top, has the same area at the top as at the bottom. What is the volume of the slab that stands on the r^{th} plane, including the additions? Its height is $\frac{1}{n} H$ and so its volume $\frac{r^2}{n^3} HS$. Now give an ex-

pression for the volume of the n slabs, including additions.

—The volume is $\frac{1}{n^3} HS + \frac{2^2}{n^3} HS + \dots + \frac{n^2}{n^3} HS$, or more compactly written $\frac{HS}{n^3} (1 + 2^2 + 3^2 + \dots + n^2)$.

31. The expression is not yet in a usable form. We can put it into a better form by a method which seems at this stage a mere trick, but which you will perhaps see later to be an application of the Integral Calculus. Multiply out $(r+1)^3 - r^3$. In the equation $(r+1)^3 - r^3 = 3r^2 + 3r + 1$, put 1, 2, ... n in succession for r , and add up the left sides of the equation and add up the right sides.—We have

$$(1+1)^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1,$$

$$(2+1)^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1,$$

$$(3+1)^3 - 3^3 = 3 \times 3^2 + 3 \times 3 + 1,$$

$$\vdots$$

$$(n+1)^3 - n^3 = 3 \times n^2 + 3 \times n + 1.$$

When we proceed to add we notice that $(1+1)^3$ or 2^3 occurs in the first line and -2^3 in the second line, 3^3 occurs in the second line and -3^3 in the third, and so on. So that for the left-hand side we have nothing left but -1^3 from the first line and $(n+1)^3$ from the last, and the sum is

$$(n+1)^3 - 1.$$

The right side of each equation contains three terms. Take

the first term from each and add them. What do you get? —The sum of these terms is $3(1+2^2+3^2+\dots+n^2)$, containing the number we want to simplify. Denote this number $1+2^2+\dots+n^2$ by S_2 so that the sum of the first terms is $3S_2$. Take the second term from the right side of each equation, and find their sum.—This sum is $3(1+2+3+\dots+n)$. Call this $3S_1$. What is the sum of the third terms?—Each is 1, and there are n of them, so the sum is n .

32. You now have the equation

$$(n+1)^3 - 1 = 3S_2 + 3S_1 + n,$$

and if you had a compact expression for S_1 this would give you one for S_2 . If you repeat the work just done, but using $(r+1)^2 - r^2$ instead of $(r+1)^3 - r^3$, at what result do you arrive?—We have

$$(1+1)^2 - 1^2 = 2 \times 1 + 1,$$

$$(2+1)^2 - 2^2 = 2 \times 2 + 1,$$

$$(3+1)^2 - 3^2 = 2 \times 3 + 1,$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ (n+1)^2 - n^2 = 2 \times n + 1. \end{matrix}$$

The sum of the left sides is $(n+1)^2 - 1^2$. The sum of the first terms of the right side is $2S_1$, where S_1 means

$$1 + 2 + 3 + \dots + n$$

as before. The sum of the second terms is n . So that we have the result

$$(n+1)^2 - 1 = 2S_1 + n \quad \text{or} \quad S_1 = \frac{n^2 + n}{2} \quad \text{or} \quad = \frac{n(n+1)}{2}.$$

Put this value for S_1 in

$$(n+1)^3 - 1 = 3S_2 + 3S_1 + n,$$

and find an expression for S_2 . The equation becomes

$$n^3 + 3n^2 + 3n = 3S_2 + \frac{3n(n+1)}{2} + n,$$

$$\text{or } 3S_2 = n \left\{ n^2 + 3n + 3 - \frac{3(n+1)}{2} - 1 \right\} = \frac{1}{2} n (2n^2 + 3n + 1).$$

so that

$$S_2 = \frac{2n^3 + 3n^2 + n}{6}.$$

Ex. 5. Solve the equation $2n^2 + 3n + 1 = 0$. Use your results to express $2n^2 + 3n + 1$ as the product of two factors, and to express S_2 as the product of three factors.

33. We had the expression $\frac{HS}{n^3} (1 + 2^2 + 3^2 + \dots + n^2)$ for the volume of the pyramid together with certain small additions. And now we have an expression for $1 + 2^2 + \dots + n^2$. By means of this give a simpler expression for the volume, and by taking $n = 10, 100, 1,000,000$, see how the volume depends on the number of slabs the pyramid is cut into.

—The expression is $\frac{2n^2 + 3n + 1}{6n^2} HS$. When $n = 10$ the

multiplier of HS is $\frac{231}{600}$ or 0.385 ; when $n = 100$ it is $\frac{20801}{60000}$ or 0.33835 . When $n = 1,000,000$ the multiplier is

$$\frac{2 \times 10^{12} + 3 \times 10^6 + 1}{6 \times 10^{12}} \quad \text{or} \quad \frac{1}{3} + \frac{1}{2 \times 10^6} + \frac{1}{6 \times 10^{12}};$$

the second term affects only the sixth or seventh decimal place, so that the value $\frac{1}{3}$ is very approximate.

By putting the expression for the volume in the form

$$\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) HS$$

it is made obvious that the greater the number of slices we cut the pyramid into the more nearly $\frac{1}{3} HS$ gives the volume including the additions. How does the volume of the additions, considered by themselves, vary with the number of slices? Imagine all these additions lowered vertically till they rest on the base of the pyramid.—When resting on the base of the pyramid all the additions fit inside one another and occupy part of the volume of a

slab of the thickness of a slice of the pyramid, that is $\frac{1}{n} H$, and whose area is the base of the pyramid. So their total volume is less than $\frac{1}{n} HS$, and it diminishes as n increases. So that by increasing the number of slices we can show

that to any degree of accuracy we choose $\frac{1}{3}HS$ is the volume of the pyramid.

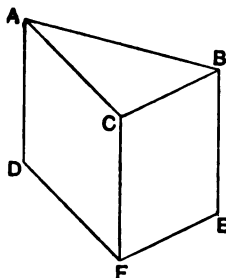


FIG. 20.

Ex. 6. Show that if two pyramids of equal base-area and equal height stand on the same plane, they are cut by any plane $\parallel$ to the bases in sections of equal area.

Deduce that pyramids of equal height and base-area have the same volume.

Ex. 7. Assuming that pyramids on the same base and of the same height have equal volumes, show how to cut up the prism of Fig. 20 (bounded by two $\parallel$ planes, and three other planes $\perp$ to them) into three pyramids equal in volume. Work with paper models. (See Fig. 21, showing the dissected prism, and note that a pyramid $CABE$ would have the same base and height as $F_3A_3B_3E_3$.)

Congruence.

34. Can two tetrahedra $ABCD$ and $A'B'C'D'$ that have the six edges equal in pairs be brought to have each pair of edges coinciding? Actual material tetrahedra would, of

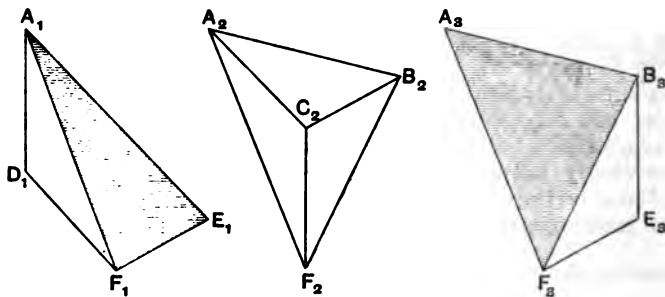


FIG. 21.

course, get in one another's way; assume that they penetrate one another as freely as ghosts.—Place A' on A and $A'B'$ along AB ; the lengths being equal B' lies on B . The pos-

sible path for C' is now a circle, the intersection of a sphere with centre A and radius AC or $A'C'$, and a sphere with centre B and radius BC or $B'C'$. And C lies on this circle; so we can rotate the second tetrahedron till C' coincides with C . This is all the fitting we can do; the tetrahedron $A'B'C'D'$ is now fixed. D' lies at an intersection of three spheres whose centres are A, B, C and radii AD, BD, CD (since these are equal to $A'D', B'D', C'D'$). There are two intersections of these three spheres and D is at one of them; so that D' may or may not coincide with D .

The tetrahedra may be equal as two right hands are (two ghostly right hands), so that they will fit together point by point, line by line, and face by face; they are then said to be congruent. Or they may be equal as a right and left hand are, or a right hand and its image in a mirror, so that though they correspond point by point, line by line, and face by face, and have corresponding lines equal in length and corresponding faces congruent in the sense of plane geometry, yet they cannot be fitted together; they are then said to be images of one another.

Ex. 8. Show that when the tetrahedra are like right and left hands, D' is the image of D in the base ABC (see the discussion of hanging the electric lamp). Show also that every point of the edge $A'D'$ is the image in the base of a point on AD .

35. Give some other ways of specifying a tetrahedron than by the lengths of the edges. If you include angles between edges and angles between faces, how many quantities have you to choose from?—We could specify the base ABC by three sides, two sides and an angle, or one side and two angles, as we saw in discussing triangles. BCD could now be specified by two sides more, or one side and one angle, or two angles. Then ABD could be specified by the length AD , or the angle ABD , or the angle between the faces BCA and BCD . The figure is thus specified by six data, as before. The figure has six edges, twelve angles between edges, and six angles between faces—twenty-four quantities to choose from; and any six independent ones will serve as a specification.

36. We saw that a plane parallel to the base of a tetrahedron cuts off a smaller tetrahedron having every edge pro-

portionally reduced from the corresponding edge of the original tetrahedron, and having every angle equal to the corresponding angle. If two tetrahedra $ABCD$ and $EFGH$ (Fig. 22) have all their edges in proportion, that is, if $EF/AB = FG/BC = \dots$, can you place them so that E lies

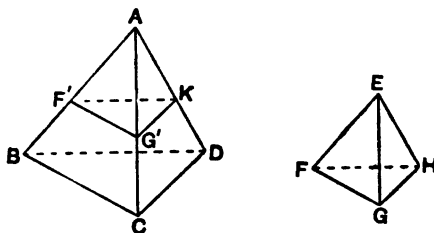


FIG. 22.

on A , while EF , EG , and EH lie on AB , AC , and AD ? Is the base FGH then $\parallel$ to BCD ? Denote the face opposite to A by a , the face opposite to G by g , and so on.—Place E on A , and EF along AB as AF' . Turn the face h or EFG about AB to lie flat on the face d or ABC , so that EG lies to the same side of AB as AC does. We know that $EF/AB = FG/BC = GE/CA$ so that the Δ s are similar, and $\angle FEG = \angle BAC$, and therefore EG lies along AC , say as AG' .

This fixes the position of the tetrahedron $EFGH$, and H lies at one of the intersections of certain three spheres. Does either of these intersections lie on AD ?—Through F' draw $F'K \parallel BD$, meeting AD in K . Since $AB/EF = BD/FH = DA/HE$, Δ s c and g are similar; and since $F'K \parallel BD$, Δ s c and $AF'K$ are similar. Hence Δ s g and $AF'K$ are similar; and having $EF = AF'$, they are equal. Thus $KA = HE$ and $KF' = HF$, so that K lies on two of the spheres that fix the position of H . Again, since Δ s b and f have their sides proportional, they are similar, and $\angle CAD = \angle GEH$. So that Δ s f and $AG'K$ have $AG' = EG$, $AK = EH$, $\angle G'AK = \angle GEH$; thus the Δ s are congruent and $G'K = GH$. So that K lies on the third sphere, with centre G' and radius GH .

We shall discuss later what happens when H does not fall

at K . When H does fall at K , are the bases c and a parallel?—Draw a plane through F' $\parallel$ to the base a . It cuts face d in a line $\parallel$ to BC , that is in $F'G'$; and it cuts face c in a line $\parallel$ to BD , that is in $F'K$. So that the base c takes up a position $\parallel$ to base a .

Images.

37. Two points A and a being images of one another in a plane, and two other points B and b images of one another in the same plane, compare the distances AB and ab .

—Let Aa meet the plane in C , Bb in D (Fig. 23). By the meaning of image $CA = Ca$ and $DB = Db$; and Aa and Bb being $\perp$ to the plane, they are $\perp$ to CD and are in the same plane. Suppose the plane figure $ACDB$ rotated about CD till it lies flat on $aCDB$. Since both $\angle ACD$ and $\angle aCD$ are right, and $CA = Ca$, CA must lie along Ca and coincides with it. So does DB with Db . Thus AB coincides with ab and is equal to it.

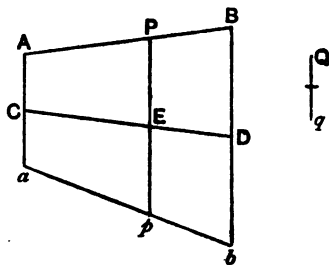


FIG. 23.

Is the straight line ab the image of the straight line AB , or is the image of AB perhaps curved?—Take a point P on AB . To find its image we suppose a $\perp$ PE drawn to the plane, and produce it its own length to p . This $\perp$, being $\parallel$ to AC , is in the same plane with AC , namely, in the plane containing AC and APB . This plane contains CD and ab , so that PE cuts these lines, say in E and p' . Also the rotation of $ACDB$ about CD brings P to the point p' , and so $Ep' = EP$. Therefore p' is p , the image of P . And the straight lines ab and AB are images of one another.

38. Take any triangle ABQ and its image abq (Fig. 23*). Are the angles AQB and aqb equal?—The Δ s have sides equal in pairs and so are congruent, and $\angle AQB = \angle aqb$.

Is the angle contained by two lines equal to the angle contained by their images?—Join two points on

* The figure shows the vertices of the Δ s, but not all the sides.

the legs of the first angle and make a triangle. We thus reduce the problem to that just discussed, a Δ and its image.

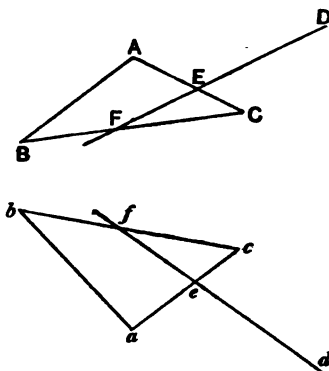


FIG. 24.

cutting two of the former lines in E and F . The images of $D E F$ are three points $d e f$ in a straight line, and since e and f lie in the plane abc , so also does d . Thus every point in the plane ABC has its image in the plane abc .

If we take the image of every point of a plane, do these images form another plane?—Suppose we fix the positions of the first plane by three points in it, $A B C$ (Fig. 24). Then we know that the images of these points and the lines joining them are three points $a b c$ and the lines joining them. Take D , any other point in the plane ABC . Draw a line through it,

39. Is the angle between two planes equal to the angle between their images?

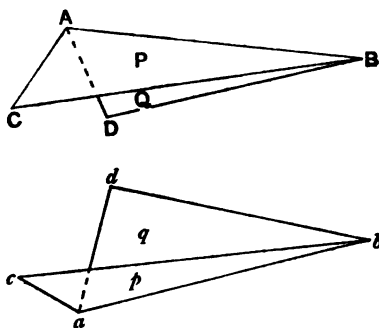


FIG. 25.

—Let AB (Fig. 25) be the intersection of the planes P and Q , AC and AD lines in the two planes $\perp$ to AB , so that $\angle CAD$ is the angle between the planes. Take the images $a b c d$ of these four points. Then the planes abc and abd are the images p and q of the planes P and Q . The Δs ABC and abc are congruent, so that $\angle cab$ is $= \angle CAB$ and

is right. In the same way $\angle dab$ is right, so that $\angle cad$ is

the angle between the planes p and q . The Δ s CAD and cad are congruent, so that $\angle CAD = \angle cad$, that is $\angle PQ = \angle pq$.

Similarity.

40. Return now to the tetrahedra $ABCD$ and $EFGH$ with all their edges proportional (art. 36). Take D' , the image of D in the face d , and compare the tetrahedra $ABCD$ and $ABCD'$ (Fig. 26).—The previous discussion shows that (A, B, C being all their own images in the face d) the two figures have all their corresponding sides and angles equal.

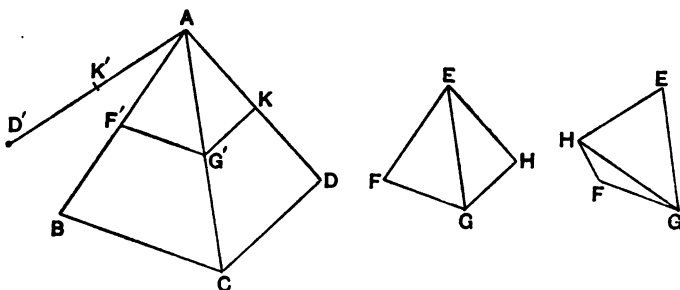


FIG. 26.

Consider again where H may lie when the face h is applied to $AF'G'$.—We saw that a point K on AD is one of the intersections of the three spheres that determine the position of H . $ABCD'$ being the image of $ABCD$, the same discussion shows also that a point K' on AD' is the other intersection of the three spheres. It shows further that $F'G'K'$ is $\parallel$ to BCD' .

Thus it is seen that if two tetrahedra $ABCD$ and $EFGH$ have their edges proportional, the smaller can be applied to the greater, or to an image of it, so as to fit it as if it was simply a part cut from it by a plane parallel to its base. When the smaller tetrahedron fits the greater like a piece cut off, when it is simply a smaller copy of the greater, they are said to be similar. In the other case, one tetrahedron is similar to the image of the other.

41. Suppose a figure of five points $ABCDE$ joined by lines

and specified by the sides of the $\triangle ABC$, the distances of D from the points $A B C$, and the distances of E from the points $A B C$. Suppose a second figure $abcde$ specified by corresponding lengths, every pair of corresponding lengths being equal. Are the two figures either congruent, or images one of the other?—We apply $\triangle abc$ to $\triangle ABC$. Then d may lie at D or at its image D' , and e may lie at E or at its image E' . If d coincides with D and e with E , the figures are congruent. If d and e coincide with D' and E' , one figure is the image of the other. But if d coincides with D and e with E' , one figure is neither congruent to, nor the image of, the other.

42. Give the corresponding properties for two such 5-point figures that have the nine lengths in question proportional instead of equal.—One figure may be similar to the other, or similar to its image, or it may be that neither relation holds.

Can you suggest a method of specification that will exclude the third possibility?—If we choose the $\triangle ABC$ so that D and E lie on the same side of it, and impose the condition that d and e are to lie on the same side of $\triangle abc$, $abcde$ must be similar either to $ABCDE$ or to its image.

Can you suggest a specification that will make the figures similar?—Take an ordinary right-handed corkscrew or screw-nail. Suppose our figure solid, so that it will hold a corkscrew or screw-nail, and bounded by the plane ABC . Screw the instrument into the plane ABC at right angles, naming the points $A B C$ in such an order that the direction of rotation is from A to B , from B to C , and from C to A (Fig. 27). The screw is at the same time going through the plane towards D and E . Make it a condition that a screw screwed through the plane abc in this way should also go through the plane towards d and e . Then, when we come to fit the figures together, there is no choice of position for d and e , and the figures must be similar.

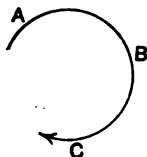


FIG. 27.

Circle and Sphere.

43. Consider in particular two figures of frequent occurrence, the circle and the sphere, and let us try to find expressions for certain lengths, areas, and volumes.

Suppose two circles of different radii drawn, and radii $OA \ OB \dots oa \ ob \dots$ drawn in each at intervals of a degree (sketch in Fig. 28). Join $AB \ BC \dots ab \ bc \dots$. The figure $ABC \dots$, a plane figure bounded by a number of straight lines, is called a polygon, and the distance it measures round, $AB + BC + \dots$, is called its perimeter. Compare

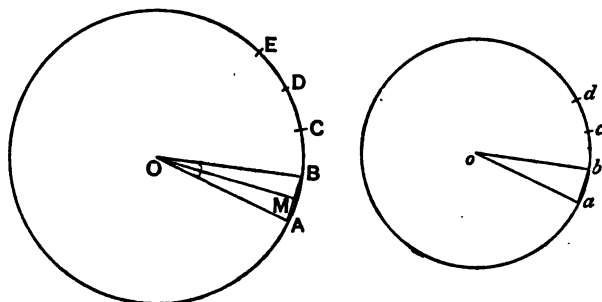


FIG. 28.

the $\Delta s \ OAB$ and oab , and compare the perimeters of the two polygons $ABC \dots$ and $abc \dots$.—The angles AOB and aob are equal as each is 1 degree, and $AO/BO = ao/bo$ as each ratio is unity; so that the triangles AOB and aob are similar. And calling the two radii R and r , $AB/R = ab/r$. Also there are 360 triangles, $OAB \ OBC \dots$, all congruent, so that the perimeter P of the polygon is $360 \times AB$; the perimeter p of the second polygon is $360 \times ab$. Hence $P/R = p/r$.

44. The perimeter of the polygon $ABC \dots$ is for most purposes the same as the perimeter or circumference of the circle. But if we want a polygon that will fit the circle closer we may suppose the angle AOB a hundredth or a millionth of a degree, or even less. What relation have we then?—Just as before $P/R = p/r$, and the smaller we make

the angles AOB and aob the more exactly do P and p become the perimeters of the circles.

The ratio of the circumference of a circle to its radius, which has been shown to be the same for all circles, may be found in various ways. You found it some time ago (chap. IV) experimentally. Calculation by methods you do not yet know gives it as accurately as we like; it is about 6.28. The ratio of the circumference to the diameter, which is half this ratio, has a special symbol π . Its value to greater accuracy than we are likely to want is 3.14159.

Ex. 9. Assuming that when x is the number of degrees in a small angle $\sin x = \frac{\pi x}{180} - \frac{1}{6} \left(\frac{\pi x}{180} \right)^3$, show that in article 43 AB is $2R \sin 0.5^\circ$, and find by how much the ratio P/R differs from 2π when the angle AOB is 1° .

45. Find expressions for the areas of the $\triangle OAB$ and the polygon $ABC\dots$, and deduce one for the area of a circle.—Draw $OM \perp$ to AB , and the area of $\triangle AOB$ is $\frac{1}{2} OM \times AB$. The polygon contains 360 congruent triangles, so that its area is $\frac{1}{2} OM \times AB \times 360$ or $\frac{1}{2} OM \times P$. The smaller the angle AOB the more nearly does OM become equal to R , the more nearly does P become the perimeter of the circle, and the more nearly does the polygon coincide with the circle; so that the area of the circle is $\frac{1}{2} R \times 2\pi R$ or πR^2 .

Ex. 10. Using the expression $\frac{1}{2} OA \cdot OB \sin AOB$ for the area of $\triangle AOB$, find by how much the area divided by R^2 differs from π when the angle AOB is 1° . (See Exercise 9.)

Ex. 11. Draw radii of a circle at intervals of 18° , there being thus 20 radii or 10 diameters. Cut out the circle and cut it in two along one of the diameters. Cut each semicircle along the nine radii to the circumference, but not through it. Spread out each half-circumference as straight as you can, and in this form fit the two pieces together so as to give an area roughly rectangular. What are its length and breadth?

Ex. 12. Draw a large circle or a quarter of a circle on squared paper. By counting squares calculate
area of circle $\div$ square of radius.

46. What is the nature of the curve in which a sphere is cut by a plane through its centre?—All the points of the

curve lie in a plane and at the same distance from a point in the plane; that is, the curve is a circle.

What is the section by any other plane?—Let OM be the $\perp$ from the centre O of the sphere on the plane, P any point on the intersection of the sphere and plane (Fig. 29). OP is the radius R of the sphere, and the angle OMP is right, so that $MP^2 = R^2 - OM^2$, and the length MP is the same for all positions of P . This section is also a circle; its radius is less than that of the central section.

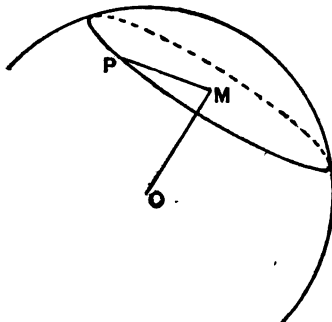


FIG. 29.

47. Let us now discuss the area of a spherical surface and the volume of the space enclosed. With the circle we showed first that the ratio of the perimeter to the radius was constant, and we might proceed similarly here. For the area of the circle we might have proved that the ratio of the area to the square on the radius is constant, and then found the value of the ratio; we did both at once by finding the expression πr^2 , which shows the ratio to be constant, and at the same time gives its value π . It is on this latter plan that we shall treat the sphere.

Suppose O the centre of a sphere, ABC the section by a plane through O (Fig. 30), OD a line $\perp$ to the plane ABC . Suppose a line, always $\parallel$ to OD and always meeting the circumference of the circle ABC , to travel round; it traces out a surface called a cylinder; OD is called the axis of the cylinder. Suppose a plane X , $\perp$ to OD , to cut the cylinder in the curve EFG and the sphere in the curve HKL . What are these curves?—We know already that the section of the sphere is a circle. The axis of the cylinder cuts the plane X in a point W at the same distance from every point of the section EFG of the cylinder, so this section is also a circle.

Suppose another plane Y , $\perp$ to OD , to cut the cylinder in

MNP and the sphere in QRS . The planes X and Y cut a strip from the cylinder and a strip from the sphere. These strips we will now compare. Call the radius of the sphere r , and the distance between the planes X and Y h , and give expressions for the length and breadth of the strip cut from the cylinder.—Suppose a plane through OD to cut the sphere in the line AQH (line not drawn in Fig. 30) and the cylinder in the line AME . The breadth EM of the strip is simply h , the distance between the planes. The length is

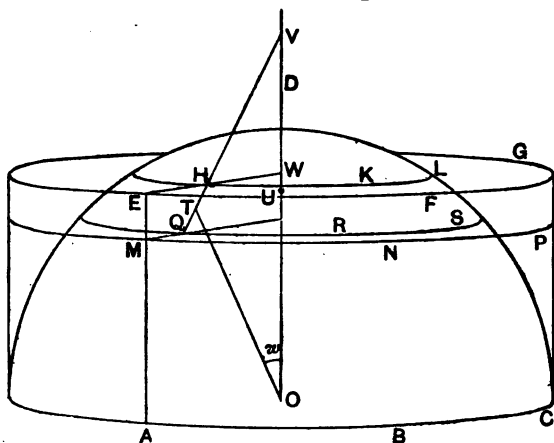


FIG. 30.

the circumference of the circle EFG or of the circle ABC ; it is $2\pi r$.

Now join QH , and from O draw $OT \perp$ to QH and therefore bisecting it. Denote the angle TOD by w . What is the relation between EM and HQ ?— $\angle EHQ = w$, so that $EM = HQ \sin w$ or $HQ = h/\sin w$.

Suppose a plane through T and $\perp$ to OD to cut OD in U . Give an expression for the length TU .— $\angle TUD$ being a right angle, $TU = TO \times \sin w = s \sin w^*$.

Now suppose $\triangle OUT$ to turn about OD , carrying QH with it. What sort of surface does QH trace out?— H moves along the circle HKL and Q along the circle QRS , so that HQ traces out a strip not unlike

* Denoting TO by s ; $s = r$ approximately when the strip is narrow.

the strip cut by the planes X and Y from the sphere. The strip cut by these planes from the sphere would be traced by the rotation of the arc QH of the circle AQH , and not the chord QH .

48. Consider the strip traced by the chord QH , and taking its area as given approximately by its breadth QH , multiplied by the length of the path traced by T , give an expression for the area; and compare with the strip cut from the cylinder. — TU being $= s \sin w$, the length of the path of T is $2\pi s \sin w$. We have seen that $QH = h/\sin w$. So that the area is $2\pi s \sin w \times \frac{h}{\sin w}$ or $2\pi sh$, or approximately $2\pi rh$.

We have seen that the strip of cylinder has a length $2\pi r$ and a breadth h , so that its area is $2\pi rh$.

In what circumstances do the areas of the strips traced by the arc QH and the chord QH differ greatly, and when little? — When Q and H are far apart, the arc and the chord separate widely about the middle, and a comparison of the areas traced by them is not easy. But when Q and H are close together the curvature of the arc is slight, and the strips pretty nearly coincide. For instance, if the sphere you were considering was the Earth, you could take the strips a mile wide, that is, take Q and H a mile apart, without finding a sensible difference between the arc and the chord and between the corresponding strips.

Let us consider more closely the area of the strip traced by the chord QH . Suppose QH produced to meet the axis OD in V . How do you know QH produced will meet OD ? Consider the surface traced out by QV by rotation round the axis OD ; this surface is called a cone. Suppose a sheet of paper of the form of this cone slit along VHQ and spread out. What form has it then (Fig. 31)? Such a figure bounded by an arc of a circle and two radii is called a sector of the circle. Give an expression for the area of the sector in terms of the radius $VQ = x$ and

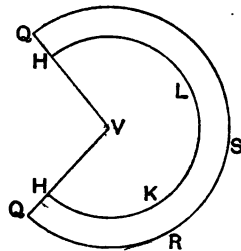


FIG. 31.

the length l of the arc.—We divide up the area by a great number of radii into pieces approximately triangular, and so find the area to be $\frac{1}{2}xl$.

Consider the cone traced out by VH , and the sector into which it spreads out. Express the area of the sector in terms of the radius $VH = y$ and the length m of the arc. Hence give the area of the strip into which the surface traced by HQ spreads out.—The area of the strip is the difference between the sectors, namely $\frac{1}{2}xl - \frac{1}{2}ym$. Show that

the two sectors are similar, and express x and y and the area of the strip in terms of l m and $HQ = t$.—The similarity

gives $\frac{x}{l} = \frac{y}{m}$, and we have $x - y = t$. These two equations

give x and y in terms of l m t ; namely, $y = \frac{mt}{l-m}$ and

$x = \frac{lt}{l-m}$. We have then for the area of the strip $\frac{l+m}{2} \times t$.

Prove now that the expression $2\pi sh$ used for the area traced by the chord QH was not merely an approximation, but was accurate.

Ex. 13. Suppose the radius of the circle QRS of Fig. 30 to be 5.6 cm. and the length VQ to be 7.8 cm. Calculate the angle of the sector into which the cone spreads out.

Draw the sector, cut it out and make the cone from it.

49. It appears then that two planes $\perp$ to OD and pretty close together cut from the sphere and the enclosing cylinder strips of approximately equal area. Can you deduce a method of finding the area of the whole spherical surface?—We conceive drawn a series of planes $\perp$ to OD and cutting the whole sphere into pretty narrow strips. Each strip is equal in area to the corresponding strip of the cylinder. So the whole area of the sphere is equal to the part of the cylinder enclosed between the two planes that bound the sphere. These planes enclose a length $2r$ of the cylinder. Slit the cylinder along a line $\parallel$ to the axis OD , and the area we want spreads into a rectangle, measuring $2\pi r$ by $2r$. Its area is therefore $4\pi r^2$.

We have shown that, approximately, a sphere of radius r cm. has a surface of area $4\pi r^2$ sq. cm.; the form of the

result showing that the ratio of the area to the square on the radius is constant, and that the constant value of the ratio is 4π or about 12.6.

The result is not proved to be accurate. The narrower we make a strip the smaller is the possible error for that strip. But the narrowing of the strips increases their number, and so (when we are discussing the whole sphere) increases the number of errors. For complete proof it is necessary to ascertain which effect is greater, the reduction of the individual error or the increase in the number of errors; but this cannot be done here. As a matter of fact the expression $4\pi r^2$ is accurate.

50. To calculate the area of a circle we divided its circumference into many parts, small enough to consider straight, and so divided the area into Δ s. Can you find the volume of a sphere in a similar way?—We divide up the surface of the sphere into a network of triangles so small that each may be considered a plane triangle. Each of these triangles forms with the centre of the sphere a tetrahedron. The height of the tetrahedron is the radius of the sphere, and its base the triangle, so that its volume is $\frac{1}{3}r$ multiplied by the area of the Δ . The total volume is therefore $\frac{1}{3}r$ multiplied by the sum of the areas of the Δ s, that is, by the area of the sphere or $4\pi r^2$. So the volume of the sphere is $\frac{1}{3}r \times 4\pi r^2$ or $\frac{4}{3}\pi r^3$.

This formula shows the ratio of the volume of the sphere to the cube on the radius to be constant, and shows this constant value to be $\frac{4}{3}\pi$ or about 4.2.

51. Assuming $4\pi r^2$ to give the area accurately, can you tell whether $\frac{4}{3}\pi r^3$ gives the volume accurately?—Consider the surface of the sphere divided up into triangles. Let us replace these triangles by the plane triangles that have the same angular points. We now have a figure bounded by plane faces. [It is called a polyhedron.] This figure can now be divided into pyramids with vertices at the centre O , and the volume of each is the base area multiplied by the distance of O from the base. These distances differ a little one from another. Suppose r_1 the greatest and r_2 the least; and suppose S the sum of the base areas. Then we know the sum of the volumes of the pyramids to lie between

$\frac{1}{3}r_1S$ and $\frac{1}{3}r_2S$. Now suppose the divisions of the spherical surface to become smaller and smaller. Our polyhedron fits the sphere more and more exactly as the divisions become smaller. The plane Δs , whose total area is S , fit the surface more and more accurately, so that S becomes more and more nearly $= 4\pi r^2$. And the distance of each plane triangle from the centre of the sphere (including r_1 and r_2) becomes more and more nearly equal to r . Therefore the expression $\frac{1}{3}r \times 4\pi r^2$ for the volume of the sphere is accurate.

Other methods of specification.

52. We have already (chap. VII) had co-ordinates as a method of giving the positions of points. Consider again

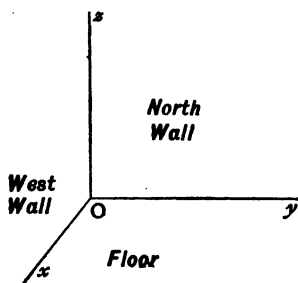


FIG. 32.

a lamp L hung from three points $A B C$ of the ceiling. Let us take as co-ordinate planes the floor, the north wall and the west wall, and as co-ordinate axes Ox running south, Oy running east, and Oz upwards (Fig. 32). The co-ordinates of a point will always be given in the same order so that to get from the origin O to the point $(1.7, 1.2, 2.0)$ a point must travel 1.7 units in the direction Ox , 1.2 in the direction

Oy , and 2.0 in the direction Oz .

The co-ordinates of the lamp being $(1.7, 1.2, 2.0)$, and the co-ordinates of the three points of suspension $(0, 0, 3.4)$, $(0, 2.2, 3.4)$, $(2.8, 1.3, 3.4)$, and the unit a metre, calculate the distances apart of the points of suspension, and the distance of the lamp from each.

53. Another method of specification is by projection on two planes at right angles. The projection of a point on a plane is the foot of the $\perp$ let fall from the point on the plane. Suppose the lamp L , the points of suspension $A B C$, and every point of the suspending strings projected

on the north wall and also on to the floor. Now suppose the wall turned about its base XY into the horizontal plane. Taking Fig. 33 to represent floor and wall and the projections

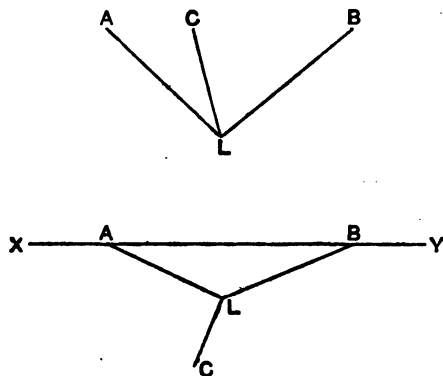


FIG. 33.

on a scale of $1/100$, find the distances apart of the points A B C , and the length of each suspending string.

A projection on a horizontal plane is called a plan, a projection on a vertical plane an elevation.

54. Volumes of similar solids. We have seen that a tetrahedron of height H and base-area S has a volume V equal to $\frac{1}{3}HS$; the units of length, area, and volume being the linear, square, and cubic centimetre, or any other corresponding set of units. Suppose a second tetrahedron to be similar to the first, the ratio of similarity being k , so that every dimension of the second tetrahedron is k times the corresponding dimension of the first. What are its height, base-area, and volume?—The height is kH , the base-area k^2S (chap. IX, art. 33), and in consequence the volume $\frac{1}{3} \times kH \times k^2S$ or k^3V .

When two tetrahedra are similar and the height of one is twice, or three times, the height of the other, show how the greater may be cut up into smaller tetrahedra each congruent with the smaller given one.

Show how to divide two similar pyramids on polygonal bases into similar tetrahedra, and compare their volumes. Show how to divide two similar solids, bounded by plane faces, into similar pyramids, and compare their volumes.

Show how to extend the result to similar solids bounded by curved surfaces. You thus arrive in another way at the result of article 85 of chapter IX, that the volumes of similar solids are as the cubes of their linear dimensions.

EXERCISES.

14. In a pyramid $OABCD$ the base $ABCD$ is a square a inches in the side. The centre of the base is M and the line OM is perpendicular to the diagonals AMC and BMD of the base and is b inches long. N is the middle point of AB . By considering in succession the right-angled triangles ABC , AMO , ANO , find expressions for AM , AO , ON , and the area of the face ABO , in terms of a and b .

15. If a parallel be drawn to one side of a triangle so as to cut the other two sides, show that it will divide them in the same ratio.

The volume of a cone of height h and radius of base a being $\frac{1}{3}\pi a^2 h$, find the expression for the volume of a truncated cone of height h , radius of base a , and radius of top b .

A chimney of solid brickwork is 120 feet high; the radii of its base are 12 feet external, 9 feet internal, and of its top 6 and 5. Find the volume in cubic feet.

16. A 45° set square resting on a table is turned about its hypotenuse as a hinge through an angle of 56° . Represent the set square, full size, in plan and elevation in the new position, taking the hypotenuse as 6 inches in length and parallel to the plane of elevation.

17. Determine the height of a mountain C , the following data being given:—a base line AB 3000 ft. is measured in the horizontal plane and the angles CAB and CBA are observed to be $55^\circ 20'$ and $98^\circ 18'$ respectively, and the angle of elevation of C from B is $10^\circ 7'$.

18. The two faces of a burning glass are parts of spheres of radii a and b , and the diameter of the glass is c . Find an expression for the thickness of the glass at the centre.

19. A tent is supported by two upright posts with a bar fastened across them at the top. Discuss how many ropes should be used

to hold this frame in position, and how the ropes should be placed.

20. A drawing board $ABCD$ slopes at an angle of 35° to the horizontal; the lower side BC is horizontal, and O is a point in it; OP is 10 inches long, lies on the drawing board, and makes an angle of 40° with BC . Find the vertical height of P above BC , and deduce the slope of OP .

21. The plane of a coal seam dips at 22° to the horizontal, and the thickness of the seam measured vertically is 5.4 feet. Calculate the thickness measured at right angles to the faces of the seam.

22. An excavation is made in the side of a hill of regular slope, the bottom being horizontal and 54 feet square, and the sides vertical. One corner of the bottom is on the original ground surface, the adjacent ones are 4 feet and 2 feet below the original surface, and the opposite corner 6 feet below. How many cubic feet are removed?

Show that there is a relation between the depths of the four corners, so that it would be sufficient to give the depths of three. And find an expression for the volume removed in terms of the depths of the corners.

23. Show that if three planes meet so that the three angles between their lines of intersection are right angles, then the three planes are perpendicular to one another.

If OA , OB , OC are the three lines of intersection of three planes, and the angles AOB and BOC are right, while $\angle COA = 55^\circ$, what are the three angles between the planes?

24. The line A of steepest slope up the plane face of a hill rises 28 in 100 (28 vertical, 100 horizontal). Determine graphically the slope of a path B which makes with A an angle of 40° . Give your answer as so many in 100, stating your result to the nearest integer.

25. To determine the distance between two points, P and Q , observations are taken from two points, A and B , 400 feet apart; the four points not being in one plane. It is found that

$$\begin{aligned}\angle PAQ &= 87^\circ, \angle PAB = 94^\circ, \angle QAB = 25^\circ, \\ \angle PBA &= 53^\circ, \angle QBA = 129^\circ.\end{aligned}$$

Find, by drawing or by calculation, the distance between P and Q .

26. Four points A , B , C , D are in the same plane; A and B inaccessible, and lying on the same side of CD . Their distance apart is to be determined from the following observations:

$$CD = 200 \text{ ft.}, \angle CDB = 80^\circ, \angle DCA = 120^\circ, \angle ACB = 65^\circ, \angle ADB = 60^\circ.$$

Determine the distance AB by drawing and by calculation.

27. How many degrees of freedom have (1) a triangle of jointed rods, not confined to one plane, one side held; (2) a jointed quadrilateral, confined to one plane, one side held; (3) the same quadrilateral when not confined to one plane; (4) a flail when the end of the handle is fixed; (5) a doll fitted with hinge joints at the hips and shoulders; (6) a doll fitted with ball and socket joints at hips and shoulders?

28. A straight bar 2 feet long is suspended horizontally by two vertical strings, each 2 feet long, attached to its ends. The bar is twisted round its centre, the strings being kept tight, and the bar horizontal, till the centre is raised a foot. Through what angle is the bar twisted?

29. The following statements contain obvious errors; show how to detect them with the least possible work. You are not to rectify them but to indicate why you consider them wrong. The last figure in each result is to be regarded as approximate,

(a) $(0.339697)^2 = 1.153937$.

(b) The side of a regular pentagon* inscribed within a circle of diameter 4.7 in. is 2.34 in. long.

(c) The area of a triangle with sides 10 cm., 14 cm., 15 cm. is 98 sq. cm.

(d) $\frac{27^2 \times 7 \times 23}{8 \times 22 \times 277} = 24.07$,

(e) If one root of the equation $x^2 - 10x + 17 = 0$ is 2.17, the other must be 7.43.

30. A garden 1,000 square yards in area is watered with a hose that delivers 5 gallons a minute. How long will it take to deliver water that would cover the garden to a depth of a quarter of an inch if it did not run away? A gallon is 277 cubic inches.

31. The amount of water which can be carried off by a circular drain pipe, whose diameter is D feet, is $\frac{A \times D^3}{C + \sqrt{D}}$ cubic feet per minute, where $A = 454.3$ and $C = 1.005$. Find, with as little work as possible, whether a drain pipe 21 inches in diameter will suffice to carry off 2,000 cubic feet per minute.

Find in miles an hour the average speed of the water in the pipe, the area of the cross-section of the pipe being $0.785 \times D^2$ square feet.

* A pentagon is a rectilinear plane figure with 5 sides, a polygon is a rectilinear plane figure with any number of sides. A pentagon, or any polygon, is called regular when all its sides are equal and all its angles are equal.

32. Figure 34 shows a road that is to be made. The breadth of each part is marked on the figure, and the length CE is also given; all these in links. The bearings of various lines are given, by the

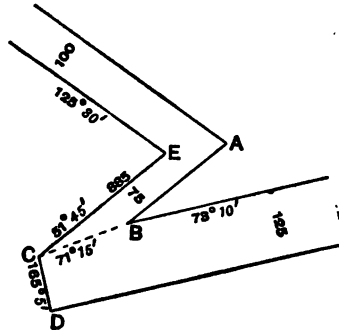


FIG. 34.

angles the lines make with the north measured round to the east. From these data draw the figure to scale and see if there are data enough and not more than enough. Then calculate the lengths AB and CD .

ADDITIONAL EXERCISES

Length, Area and Volume.

1. Compute to two significant figures, the diameter and cross sectional area of a specimen of copper wire, one mile of which weighs 4 lb. One cubic inch of copper weighs 0.32 lb.

2. A reservoir, with vertical walls, has an area of 1,432 square yards. It is supplied by a pipe, 3 square feet in section, through which water flows at the rate of 1 mile an hour. Find how much the water will rise in the reservoir in one hour.

3. Calculate the weight of a cast-iron pipe from the following dimensions:—Inside diameter 15 inches, thickness 0.8 inch, and length 8 ft. Density of metal = 465 lb. per cu. ft. It is assumed that there are no flanges or other enlargements at the ends.

It is required to coat a mile of such pipe inside and outside with some protective material at the rate of 20 sq. ft. for 7d. Calculate the cost.

4. It is desired to dig a trench 4 feet wide with vertical sides and horizontal base across a piece of ground, the contour of which is given by the following measurements. The bottom of the trench is the datum line. Calculate the volume of earth excavated in cubic feet.

Horizontal measurements along datum line in feet.	0	5	13	21	28	33	40	45	52
Height of surface above datum line in feet.	0	5	8	10	7	5	8	5	0

5. A river basin of a square miles receives a rainfall of b inches per 24 hours, half of which drains into the river; whose section, where it leaves the district, is c square yards. Find a formula for the velocity of the river at this point, in feet per second, supposing the whole process to be steady, and the speed of the water in the river to be the same at all depths.

Give a numerical value to your answer when $a = 180$, $b = 1/8$, $c = 160$.

6. Draw a circle 5 inches in diameter. This represents the end of a log of timber to a scale of half size. Inscribe in it any rectangle whose width may be called b and depth d . If this rectangle represents a possible joist which may be sawn out of the log, its strength M is given by $M = 300bd^2$. Find M for six or eight possible rectangular joists which may be cut out of the above log. Then by plotting M vertically and b horizontally determine what width (b) will produce the strongest joist.

7. $ABCD$ is a rectangle of paper whose sides are 3 inches and 2 inches long. Draw such a rectangle and the four creases that could be made by folding it so that a short side lay along a long side.

Prove that the intersections of these four creases with one another are the corners of a square, and that the area of this square is half a square inch.

8. Draw a right-angled triangle with equal sides and with hypotenuse equal to 1.3 inches. On the other side of the hypotenuse construct a square.

Compare the area of the triangle with that of the square.

If the figure thus drawn represents the end-view of a haystack whose breadth is b feet and length s feet, what is the volume of the stack in cubic feet?

9. Find the area of the field drawn in Fig. 1 to a scale of 1 inch to 120 yards; copy the figure, and from the result of your calculation or otherwise draw a line parallel to ab that will divide the figure into two equal parts.

If df is bisected in g and ac in h , give reasons for thinking that a straight line through g and h divides the figure into two equal parts.

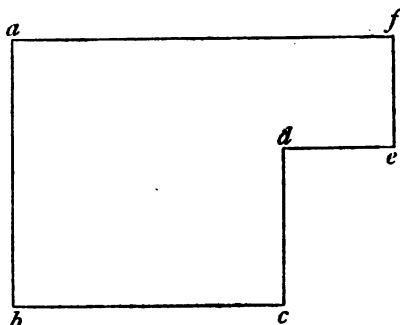


FIG. 1.

10. A swimming-bath has length l metres, breadth b metres, depth at shallow end h metres, and at deep end $h+k$ metres. Find expressions (in litres) for the quantity of water in it when the water is x metres deep at the deep end, distinguishing the cases when the sloping bottom is covered and when it is not.

For the case in which $l = 14.73$, $b = 7.88$, $k = 0.84$, and $k = 1.04$, draw a graph showing the quantity of water for various depths (scales:—depth, 4 inches to 1 metre; volume, 1 inch to 20,000 litres). Show how this graph enables you to tell the depth of the water at any time when the bath is being filled, if the rate of inflow of water is given; and, taking the rate of inflow as 50,000 litres an hour, find the depth after half an hour, after 1 hour, and after $1\frac{1}{2}$ hours.

11. A man playing 5 holes of a golf course walks first 280 yds. due east, then 140 yds. 20° south of east, then 300 yds. due south, then 200 yards 40° west of north, then 220 yds. 30° west of south, thus arriving at the fifth hole. Draw the plan of the course to a scale of 100 yds. to the inch, and find from it how far the fifth hole is from the start.

12. Draw a circle of radius 2 inches. Take a diameter AB , and divide it into ten equal parts. Call the points of division $C_1, C_2, \dots, C_9$. Draw the set of chords $D_1C_1E_1, D_2C_2E_2, \dots$ at right angles to AB , meeting the circumference in D_1, E_1, D_2, E_2 , &c. F_1 and G_1 are the feet of the perpendiculars let fall from C_1 on AD_1 and AE_1 respectively. Similarly let fall perpendiculars C_2F_2 and C_2G_2 on AD_2 and AE_2 , and so on.

Draw a fair curve through all the F and G points thus obtained. Measure the greatest breadth of the curve and write down your result. How would you obtain more points on the curve if required? Where do you think the curve cuts the line AB ?

13. A ladder 20 metres long is set against a house, the bottom of the ladder being 2 metres from the wall, and a man goes 14 metres up it. Then the ladder slips. Draw on a scale of $1/100$ the path of the man as the ladder slips.

14. Draw a horizontal line ABC from left to right, BC representing 50 feet to a convenient scale. Above A is a mound on which is situated a tower whose foot we will call D and top E .

It is observed that the angle $ABD = 20^\circ$, the angle $ACD = 15^\circ$, and the angle $ACE = 25^\circ$. Draw this figure to scale as accurately as you can, and measure from your drawing the height of the mound AD and the height of the tower DE .

Check your results trigonometrically, that is, by calculation.

15. The wheel in Fig. 2 is 5 inches in diameter and makes one revolution per second. On the knob A , which is fixed to the wheel, rests a door 15 inches high, free to slide between vertical guides. Show in a graph variations during 2 seconds in the height of the centre C of the door above the bottom of the wheel; represent the height full size, and a second by 4 inches.

Give an expression for the height of C in terms of the position of A .

Show what change would be necessary in the graph if the door rested on the hub B all the time the knob A was below B .

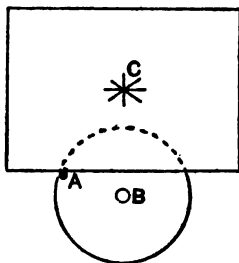


FIG. 2.

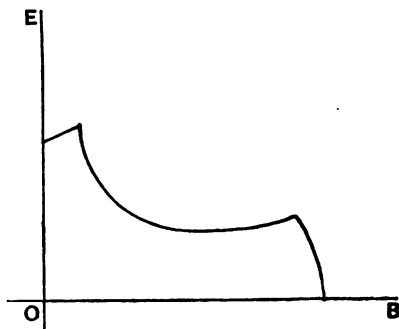


FIG. 3.

16. Fig. 3 shows a quarter of a design which is symmetrical with respect to the lines OB and OE . Prick off the figure and complete the design.

17. $ABCD$ (Fig. 4) is a frame of four jointed rods, standing in a vertical plane on a table. The rod AB is fixed to the table. Draw the frame from the given dimensions, and show by thick lines the paths that C and D are at liberty to trace out, and indicate

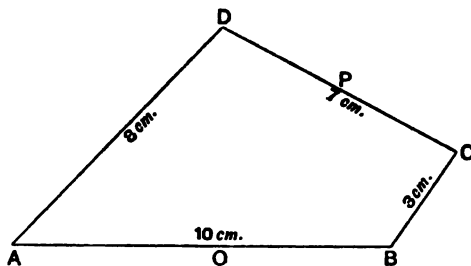


FIG. 4.

briefly the reasoning by which you determine the limits of each path.

By the help of tracing-paper, or otherwise, draw the path which P , the middle point of CD , is at liberty to trace out. Find from

the path the positions in which P is nearest to and furthest from O , the middle point of AB .

18. 'Within every parallelogram, no matter of what shape, there is a point such that all lines drawn through it from boundary to boundary are bisected at that point.' Find in any way you like the position of this point, and give a geometrical proof of the above statement.

19. Prove that parallelograms on the same base and between the same parallels are equal in area.

How could this be verified by means of a pack of cards or a pile of slates?

From this illustration deduce the corresponding property of parallelepipeds. A parallelepiped is the solid figure bounded by three pairs of parallel planes.

20. Make a quadrilateral $ABCD$, given that $AB = 4$ inches; $BC = 5$ inches; $CD = 6$ inches; $\angle ABC = 90^\circ$; and $\angle BCD = 75^\circ$.

Find the area of the figure.

If the figure represents an estate on a scale of 5 inches to the mile, what is the area of the estate in acres?

21. Draw a rectangle $ABCD$ having $AB = 5$ inches and $BC = 2$ inches, and bisect its angles by lines meeting at P, Q, R, S (see rough sketch in Fig. 5). By measurement and by general reasoning, determine what kind of a quadrilateral $PQRS$ is.

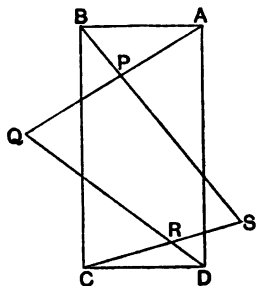


FIG. 5.

If AD were to remain constant while AB changed in value from 0 to AD , show that the area of the figure $PQRS$ (formed as before) would change in value from $\frac{1}{2}AD^2$ to 0.

22. Draw a fair-sized figure of five sides, and find its area in square inches, by reducing it to a triangle of equal area, and then finding the area of the triangle by construction.

23. Fig. 6 is a sketch of a framework made of six rods each 8 cm. long, the rods AB and CD being joined to the middle points of the other four rods. All the joints are hinged so that P and Q may be brought together or pulled apart.

Draw a straight line KL , and suppose the point P of the framework held at the point K while Q is moved to and fro along the line KL . Draw the path that E would trace out.

Prove that whatever the shape of the frame the four compartments are equal in shape and size.

How would you use another rod to lock the framework, that is, to keep its shape from changing?

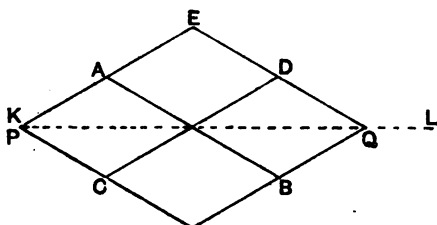


FIG. 6.

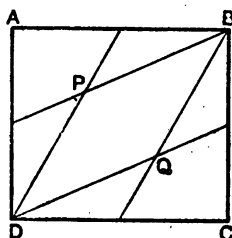


FIG. 7.

24. Draw a square 5 inches in the side (as in the sketch, Fig. 7). Join B to the middle points of DA and DC , and D to the middle points of BA and BC , and find from measurements the area of $BQDP$.

Prove that $BPDQ$ is a parallelogram and calculate its area.

25. ABC is a triangle having $AB = 3$ inches, $AC = 5$ inches, $\angle A = 90^\circ$. D is a point on BC , DE is a perpendicular let fall on AB , and DF a perpendicular let fall on AC . Draw the triangle, and from your figure fill up a table like the following:—

Length of BE	Area of rect. $AEDF$	Area of rect. $BE \times CF$
0.5 inch		
1 "		
1.5 "		
2 inches		
2.5 "		
3 "		

State and prove the relation that would hold for a perfect figure between the rectangles $AEDF$ and $BE \times CF$, whatever the lengths of AB and AC , and whatever the position of D .

Angles.

26. Take two points A and B 12 cm. apart to represent two points on a coast. From A and B four successive positions of a ship S are observed as follows :

- (1) $SAB = 96^\circ$, $SBA = 38^\circ$; (2) $SAB = 65^\circ$, $SBA = 30^\circ$;
 (3) $SAB = 35^\circ$, $SBA = 33^\circ$; (4) $SAB = 33^\circ$, $SBA = 81^\circ$.

(a) Plot the positions of the ship. (b) Assuming that the ship's course is composed of two straight lines, find how near she approached the shore. (c) Assuming that she sails equally near the wind on each tack, draw a line showing the direction of the wind.

27. AOB is an acute angle; from a point P , either within or without the angle, the line PQ is drawn perpendicular to OA and the line PR perpendicular to OB . Show that the angle QPR is either equal or supplementary* to the angle AOB . Prove both cases.

State and prove any one proposition you have made use of.

28. $ABCD$ is a rectangular piece of paper, 9 cm. by 5 cm. FE

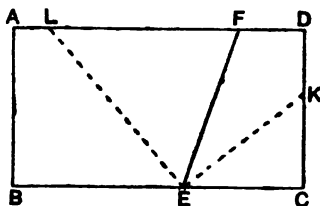


FIG. 8.

is a line drawn across it (see Fig. 8), $FD = 2$ cm., and angle $AFE = 70^\circ$. EK and EL are creases formed by folding EC and EB against EF . Draw a figure to represent the piece of paper full size, and in each angle of the figure $EKDL$ write its value.

Prove that whatever angle EF makes with AD , two angles of the figure $EKDL$ are supplementary, so long as the creased

lines cut the sides CD and AD .

29. AB and AC (Fig. 9) are two plane mirrors, and DE and GH are two parallel rays of light falling on them and reflected in the directions EF and HK , so that the angle $AED =$ the angle BEF and the angle $AHG =$ the angle CHK . Prove that (1) the angle between the mirrors = the sum of the angles AED and AHG ; (2) the angle between the two reflected rays EF and HK is twice the angle between the mirrors.

* Two angles that together make up 180° are called supplementary.

30. In Fig. 10 $OADB$ is a square, and $OAPB$ a quadrant. Think of two lines each moving uniformly and both starting together from

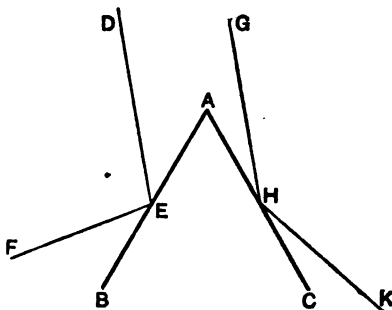


FIG. 9.

the position OA ; one, MN , moving parallel to OA , and reaching its final position BD , in the same time that the other, OP , takes to revolve about O to its final position OB . Their point of intersection C traces out a curve. Obtain this curve for a quadrant of radius 9 cm. Give the value of the angle AOC when MN has moved (a) 3.2 cm., (b) 6.4 cm., and compare these angles. Copy

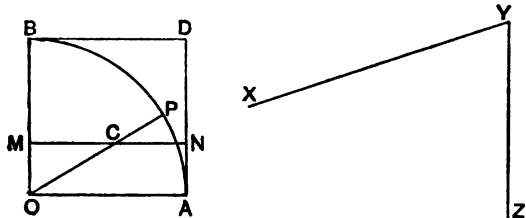


FIG. 10.

the angle XYZ , use the curve for dividing it into two parts in the ratio of 2 to 3, and state your method; the angles may be transferred by means of tracing-paper.

Triangles.

31. Make as many of the following triangles as you can. In cases where you cannot, state a property of triangles to explain the impossibility.

- (a) Sides, 5 cm., 5 cm., 9 cm.
- (b) Sides, 5 cm., 4 cm., 10 cm.

(c) Angles, 60° , 40° , 90° .

(d) Angles, 60° , 20° , 100° .

(e) One angle, 50° , side opposite 8 cm., another side 7 cm.

32. Take three points P , Q , R , and suppose them the mid points of the sides of a triangle. Construct the triangle and prove the theorem on which your construction depends.

33. $ABCD$ (Fig. 11) represents a rectangular sheet of paper the corner D of which is to be turned down so that the crease line MN shall make an angle of 30° with AD . Draw a fairly large figure showing

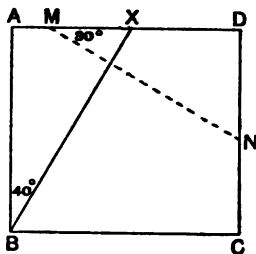


FIG. 11.

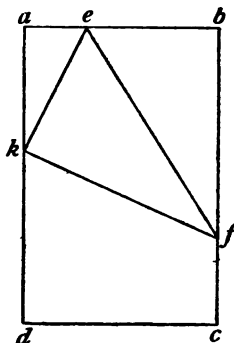


FIG. 12.

the position of MDN when turned down, and indicate how d , the new position of D , is determined by drawing. If BX is now drawn at 40° to AB and so as to cut Md , state, with reasons, what angle it must make with Md .

34. In Fig. 12 $abcd$ is a rough drawing of a sheet of paper, 4.5 inches by 7 inches. The corner b is folded down to meet the side ad in k in such a way that the creased line ef passes through e , which is 1.5 inches from a . Give a careful drawing half-size; show the exact position of the part turned down, and find by measurement the length of ef and the size of the angle kef .

35. ABC and $A'B'C'$ are two different positions of a paper triangle in the same plane and with the same side of the paper upwards. Find a point O which is equidistant from A and A' and also from B and B' . Join O to A , A' , B , B' , C , C' , and then prove that the angles AOA' , BOB' and COC' are all equal, and that O is equidistant from C and C' .

36. Fig. 13 is a rough sketch of a triangle ABC drawn on tracing-paper. The tracing-paper is pinned down by a pin at P , and then rotated through a right angle in the direction of the arrow. Draw a figure to the dimensions given, showing both the original triangle ABC and its new position abc . State how you determine the points a b c , and show what angle ac makes with AC .

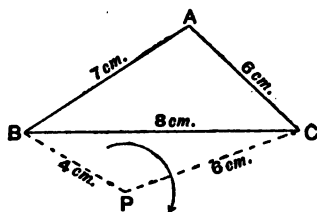


FIG. 13.

37. A triangle ABC is cut out of paper to the dimensions shown by the rough sketch in Fig. 14. The corner A is turned down so that A falls on the line BC and the crease line KL makes an angle of 30° with AB . Sketch a large figure (which need not be to scale), calculate and fill in the values of all the angles. Then draw the triangle ABC to the given dimensions, obtaining by geometrical construction the position of the crease KL and of the point a . State your method and on what geometrical principle you rely.

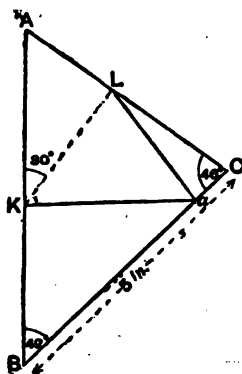


FIG. 14.

38. AB is a straight canal, C a cottage, and T a trough. CA and TB are 1 s drawn from C and T to the canal, $AB = 40$ yards, $AC = 31$ yards, $BT = 12$ yards. A man has to take a bucket from C , fill it from the canal, and carry it to T . Find by drawing, to a scale of 1 inch to 10 yards, how far the man walks if he fills the bucket at A ; if at B ; and how far for five or six other points on the canal.

At each point you have taken on the canal draw a perpendicular representing on the same scale the distance the man walks. Through the ends of these perpendiculars draw a smooth curve, and from this curve, or otherwise, find the point at which the bucket should be filled to make the journey from C to T as short as possible.

Determine this point also by geometrical reasoning.

Specification of Figures.

39. One side AB of a field $ABCD$ runs from east to west (A east of B), and is 150 yards long; AD runs north-west and is 40 yards

1000

long, and BC runs north-east, and is 90 yards long. Draw an accurate plan of the field (stating what scale you adopt), measure the length of CD , and find the area of the field in acres.

There is a spring in the field, distant 85 yards from A and 35 yards from C . Find its position on the plan (explaining your method), and measure its distances from B and D .

40. Prove two triangles equal in all respects when the three sides of the one are known to be equal to the three sides of the other.

In Fig. 15, assuming that the lines are all unequal, and that AB is parallel to DC , and AE parallel to BD , state what is the least

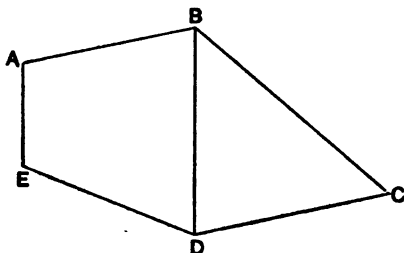


FIG. 15.

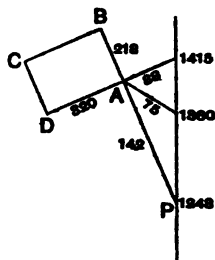


FIG. 16.

number of measurements that must be made in order that the figure may be reconstructible from these measurements alone without alteration of shape or size.

Name any two wholly distinct sets of measurements, either of which would suffice.

41. A surveyor gives the following dimensions of a four-sided field $ABCD$:— $AB = 1130$ links, $BC = 1370$, $CD = 1440$, $DA = 1040$, $BD = 1960$, $CA = 1470$, $\angle CAD = 67^\circ 30'$. Draw the field without using the values given for BD and CA , and state the steps of your drawing. Check your work by measuring BD and CA .

42. Two sides of a triangle are 3 inches and 4 inches long, and one angle is 30° . Sketch figures, with dimensions marked, showing that four different triangles answer to this description. State how you would determine which of these four has the greatest area, and with help of your instruments find approximately what this area is.

43. A surveyor fixes the position of a rectangular building $ABCD$, measuring 320 links by 218 links, as follows:—When he has gone 1,248 links along a straight road to P (Fig. 16), he is in

line with the shorter side AB and A is 142 links away. When he has gone 1,360 links, A is 75 links away; and when he has gone 1,415 links A is 88 links away and he is in line with the long side AD of the building. Take a line with a point P on it to represent the road and the point P , and draw the building on a scale of 1 inch to 100 links, stating how you do so.

Do you need all the data given? How many could you do without? Use these extra data to check the accuracy of your work.

44. Two quadrilateral figures $ABCD$ and $PQRS$ have $AB = PQ$, $BC = QR$, $CD = RS$, and $DA = SP$. Are the quadrilaterals necessarily equal in all respects? If so, give the proof; if not, add any fifth hypothesis you think necessary, and then prove the equality of the figures.

45. Criticize the proposition that three measurements are sufficient and necessary to determine a triangle uniquely in shape and size.

46. You are to obtain copies of the circle and regular hexagon in Figs. 17 and 18 by pricking through the least possible number

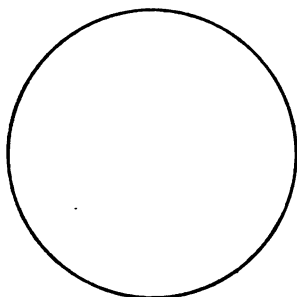


FIG. 17.

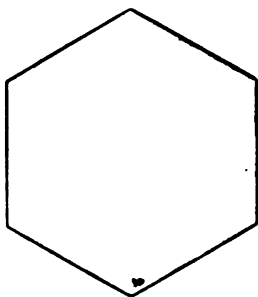


FIG. 18.

of points that will enable you to complete the figure with your instruments. State your method in each case, and give the geometrical principles you make use of for the completion of your figures.

Algebraic Symbols.

47. a men are each x feet high, b are y feet, and c are z feet. Give their average height.

48. A certain sum of money contains a pounds b shillings and c

pence, where $a < c$. A second sum of money contains c pounds b shillings and a pence. Express in pounds, shillings and pence the difference between the two sums.

Calling this difference x pounds y shillings and z pence, the sum of money obtained by again interchanging the pounds and pence is, of course, z pounds y shillings and x pence. Prove that when the last two sums are added together the result is 12*l.* 18*s.* 11*d.* Verify by a numerical example.

49. A and B travel along the same road in the same direction, A at a miles and B at b miles an hour. When A is at a point O , B is m miles ahead. If B is overtaken H hours later at a place M miles from O , find H and M in terms of a , b , and m .

Also give the values of H and M when $a = 6$, $b = 3$, $m = 9$.

50. A cyclist can travel q miles an hour on a still day, but his speed is decreased m miles an hour against a certain wind, and increased n miles an hour with this wind. How long would it take him to go 20 miles and to return, going against the wind and returning with it?

Give the time to the nearest hour when $q = 9$, $m = 3$, $n = 2$.

Taking $q = 9$, draw a graph to show the time he would take to ride 20 miles against the wind for values of m between 0 and 8. For the time represent an hour by 1 cm., and for the speed represent one mile per hour by 2 cm.

51. On the Northern Railway of France, passengers are allowed 30 kilograms of luggage free, and excess luggage is charged at the rate of 1 centime per 10 kilograms per kilometre. A tax of 10 centimes is also charged for registration. Give a formula for the amount in shillings charged for carrying p lb. weight of luggage m miles (if this weight exceeds the free limit); given 1 shilling = 125 centimes, 1 kilogram = 2.2 lb., 5 miles = 8 kilometres. Calculate from this formula the charge for conveying 100 lb. of luggage from Paris to Calais, the distance being 185 miles.

52. The exposure required to take a photograph is proportional to the square of the stop-number and inversely proportional to the speed-number of the plate. It is found that $1\frac{1}{4}$ secs. is the correct exposure with an ordinary plate (speed 80) using a stop numbered 16. What would be the exposure with a fast plate (speed 350) when the stop-number is 6.8? Express your answer in the form $1/n$ of a second, giving n to the nearest whole number.

What would be the exposure with a plate of speed number M and a stop numbered N ?

53. On a long voyage the speed of a ship gradually decreases as the ship's bottom gets fouled. If u and v are the speeds in miles per day at the beginning and end of the journey, T the number of

days taken on the journey, and s the length of the voyage in miles, these quantities are known to be connected by the formula.

$$s = \frac{3}{2} T \frac{uv(u+v)}{u^2 + uv + v^2}.$$

Find from this the time taken to perform a voyage of 6,000 miles, the speeds of the ship at the beginning and end of the voyage being 290 and 250 miles per day.

Will the ship arrive sooner or later than if the speed had been 270 miles per day throughout the journey, and what difference would this make in the time of arriving?

54. If the time of a swing of a pendulum of length l feet is taken to be $\pi\sqrt{l/g}$ seconds, where g is 32, what is the length of a pendulum of which the time of swing is 1 second? Also, prove that a change of 2 per cent. in the length of the pendulum gives a corresponding change of 1 per cent. in the time of swing.

55. In measuring a field x yards long and y yards broad, an error of $\frac{1}{10}$ per cent. may have been made in each measurement. What is the greatest possible error in the area of the field, calculated from these measurements?

56. The watch used to time a motor race gained x seconds a day. The time of the race as given by the watch was a min. b sec. Find an expression in seconds for the true time of the race.

Similarity. Trigonometrical Terms.

57. I stand 4 feet from a lamp-post and notice that the end of my shadow just reaches a wall. How far is this point of the wall from the lamp-post, if the light is 10 feet above the ground, and my height is 5 ft. 7 in.?

58. In the following give your reasons, a 'yes' or 'no' alone being of no value:

(a) 15 tons of coal cost £16. 10s. and 9 tons cost £9. 18s. Is the price proportional to the quantity?

(b) The base of one equilateral triangle is 2 in. and its area 1.73 sq. in., the base of another is 4 in. and its area 6.93 sq. in. Are the areas proportional to the bases?

(c) £50 put out at interest amounts in two years to £54, in three years to £56, and in 5 years to £60. Is the amount proportional to the time? Is the interest proportional to the time?

59. Two straight roads intersect at an angle of 25° . Starting from the crossing I walk four miles along one road. How far am I then from the nearest point of the other?

Give the result of calculation and also of measurement from a diagram.

60. *A* signals 145 words in 23 minutes, *B* 94 words in 15 minutes, and *C* 183 words in 21 minutes. Represent the times taken by horizontal distances, 1 cm. representing 2 minutes; and the number of words signalled by vertical distances, 1 cm. representing five words. Mark the three points representing the above results.

Who is the fastest signaller? Show on the diagram the graph corresponding to a rate of six words a minute.

61. A mineshaft is 1,650 feet in length. It slopes downwards at an angle of 45° to the horizon for a certain part of its total length, say x feet, and at an angle of 35° for the rest of its length. If the total depth reached is 1,000 feet, obtain an equation for x , and hence calculate x . Draw a rough figure of the shaft.

62. A carriage is moving on a level road without any slipping of the wheels. One of the wheels of diameter 3 feet picks up a speck of mud which adheres to the rim of the wheel. On squared paper plot out to scale the curve described in the air by the speck of mud, and carefully mark the points at which the speck of mud will be when the wheel has turned through 60° , 120° , 240° , and 300° . Also in the case of 60° , check your result by actual calculation.

63. Draw up a table showing in three columns the value of $10 \sin \theta$, $10 \cos \theta$, and $8 \sin \theta + 6 \cos \theta$ for each 30° from 0° to 360° . From the table draw the graphs of $y = 10 \sin \theta$ and $y = 8 \sin \theta + 6 \cos \theta$, and from the curves determine approximately a value of θ for which $\tan \theta = 3$.

64. On a line OB of indefinite length set off OP 10 cm. long. Now imagine OP to revolve about O through an angle of 90° , but to vary in length in such a way that its length at any moment is equal to its original length multiplied by the cosine of the angle it has revolved through; thus at 60° its length will be 5 cm. Use the tables to determine the path of P , and justify any statement you can make about the locus of P .

65. Four rods are made into a jointed parallelogram $ABCD$

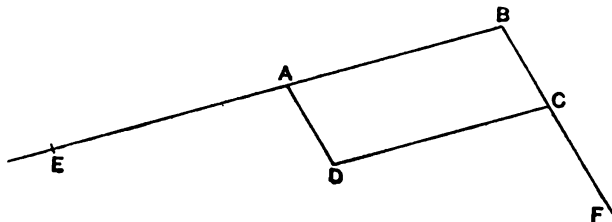


FIG. 19.

(Fig. 19). Two of the rods, BA and BC , project beyond the corners

of the parallelogram. A fixed point E on the production of BA is taken, and F is taken on BC produced so that for the shape of the parallelogram shown in the figure F is in line with E and D . Show that for all positions of the parallelogram this point F is in line with E and D .

This framework is laid on a map, F pinned down, and D is made to trace out the boundary of a country. Say how E will move. If the frame used is represented to scale by Fig. 19, what is the ratio of any line on the new map to the corresponding line on the old one?

66. The base BC of a triangle is 7 cm. long. Find the locus of the vertex when $AB/AC = 2/3$, by plotting a number of points and drawing a freehand curve through them. In the same way find the locus of the vertex when $AB + AC = 10$ cm.

One of these loci looks like a circle. Prove that it is a circle.

67. Draw a circle 10 cm. in diameter and take a point P 15 cm. from its centre. Suppose R a point on the circle and S the middle point of PR . Draw the locus of S as R moves round the circle.

Discuss, with proof, the nature of the locus of S , whatever the size of the given circle and the distance of P from it.

The Circle.

68. Find the diameter of the saucer, of which a broken fragment is shown in Fig. 20. Prick the figure off.

69. Three ships, A , B , and C , at anchor occupy the following positions with regard to a buoy P :— A is 350 yards north of P ; B is 170 yards north-east of P ; C is 420 yards east of P . Draw a plan showing the positions of the ships and the buoy, on a scale of 100 yards to an inch.

A fourth ship, D , anchors in a position equidistant from A , B , and C . Determine its position on your plan, stating its distance from the buoy.

State your method of finding the position of D , and prove that the method is correct.

70. Prick off the curves a and b (see Fig. 21), and find whether they are arcs of circles. Explain your method of testing the curves.

71. State (without proof) the construction for circumscribing a circle about a triangle.

In what kinds of triangle will the centre fall (a) inside the triangle, (b) outside the triangle, (c) on one of the sides?



FIG. 20.

You may use ruler and compasses as helps in answering the latter part of this question ; logical reasons for your answers thereto need not be given.

72. Two cog-wheels gear together ; one of them has a cogs, the other b cogs, and the cogs are set c inches apart on each wheel. When the first wheel makes one revolution, how far does a point in its rim travel ? And a point in the rim of the second wheel ? And how many revolutions does the second wheel make ?

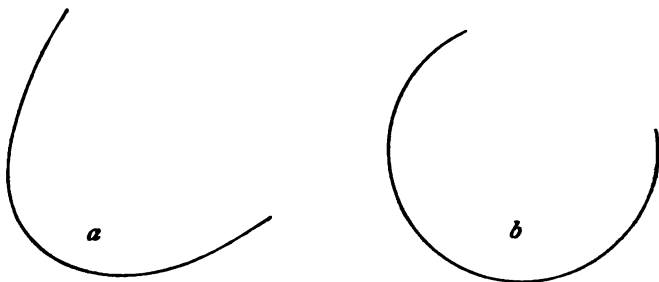


FIG. 21.

If the cog-wheels were connected by a chain instead, how many revolutions would the second wheel make for one revolution of the first ?

I wish to settle the number of teeth in the cog-wheels of a chain-driven bicycle with a rear wheel 30 inches in diameter, the bicycle to travel 6 yards for each revolution of the pedals. Plot a graph showing the number of turns of the rear wheel corresponding to 1, 2, 3, 4, &c., turns of the pedals. Deduce from your graph a convenient number of teeth for each of the two cog-wheels on which the chain works, and verify your conclusion.

73. Two circles have radii 1.3 and 2.1 inches, and their centres are 3.9 inches apart. Draw a circle of radius 1.1 inch touching them both. Can a circle of radius 0.3 inch be drawn to touch the given circles ?

74. A point P moves round the circumference of a circle whose horizontal diameter is AOB , O being the centre. Two circles are drawn on AO and OB as diameters, and OP always meets one of these circles. Show that the piece cut off OP by the circle it meets is equal to the projection of OP on the diameter AOB .

Give a graph showing (for half a revolution of P) the projection of OP as a function of the angle AOP ; that is, mark off distances on

one line to represent values of the angle, and on perpendiculars to this line mark off the corresponding values of the projection. Take OP 5 cm. long, and take 1 cm. to represent 20 degrees.

75. Fig. 22 represents a Chinese windlass. A rope supports the load D , and has its ends wrapped on two barrels of diameters 7 and 9 inches turning together on the same shaft. Another rope goes round the pulley of diameter 36 inches, which also turns with the

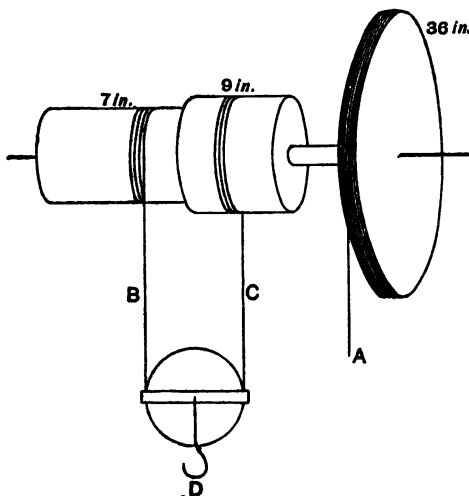


FIG. 22.

same shaft. When A descends 10 inches, what length of rope unwraps at B , what length wraps up at C , and how far does D rise?

Taking the diameters of the barrels and pulley as r s t inches, give an expression for the rise of D while A falls k inches.

76. The area of the segment contained by a circular arc APB and its chord AB , is approximately two-thirds of the area of the rectangle $ABEF$, having the same base and altitude. I form an approximate estimate of the area of a circle (Fig. 23) by measuring the area of its inscribed square, and adding the areas of the four segments outside this square as calculated by the above rule. Calculate approximately by how much per cent. the area so estimated is in excess or defect of the true area of the circle.

77. ABC and DEF are two concentric circles of radii 7 cm. and 3 cm. respectively, and AB is a chord of the outer circle distant

2 cm. from the centre. Show, both by drawing and by calculation, that AB is trisected at the points where it cuts the circumference of the inner circle.

78. Prove that if chords of a circle are drawn through a given point inside a circle, the rectangles contained by their segments are equal.

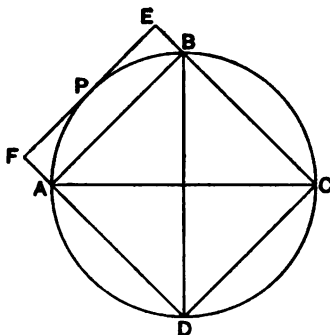


FIG. 23.

If the radius of the circle is 8 inches, and the given point 2.1 inches from the centre, calculate the value of this rectangle and verify the result from your figure.

79. A stream 20 feet wide is to be spanned by a bridge with a circular arch 7 feet high at the middle. Calculate the radius of the arch and the angle it subtends at its centre. Check your results by a drawing to scale.

80. Prove that angles in the

same segment of a circle are equal.

State (without proof) what property of a circle may be deduced at once from this theorem by conceiving the angular point to move along the circumference into coincidence with either extremity of the segment.

81. The distance of the horizon d , the radius of the Earth R , and one's height above the sea h (all in miles) are connected by the equation

$$d^2 = h^2 + 2Rh.$$

Taking the Earth's radius as 3,960 miles, find at what height the horizon is 34 miles away, and express this height in feet to the nearest foot.

82. Prove that a straight line drawn from the centre of a circle to the middle point of a chord cuts the chord perpendicularly.

P is a point external to a given circle whose centre is C . With C as centre describe a circle with radius twice that of the given circle. With P as centre and PC as radius, describe a third circle cutting the second in Q . Join CQ , cutting the first circle in R . Prove that PR is a tangent to the first circle.

83. I wish to fix the size of a circular enclosure which I may not enter; I therefore note the following points:—It is surrounded by a footpath 10 feet wide, and four lamp posts stand along the outer

boundary of the footpath at the corners of a square. Each side of this square cuts from the enclosure a strip whose greatest width is equal to the width of the path.

Find the diameter of the enclosure by calculation and also by drawing. For your drawing take a circle of radius 4 inches for the outer boundary of the footpath.

84. Find the area of the rectangle $KLMN$ (Fig. 24) inscribed in a semicircle in terms of $CK = x$ cm. and the radius $CL = r$ cm.

Taking $r = 5$, find the area for $x = 0, 1, 2, 3, 4, 5$, and give a general idea of the change in the area of the rectangle by a graph. Then determine accurately by algebra or by geometry the dimensions of the largest rectangle that could be cut from a semicircular piece of paper of radius 5 cm.

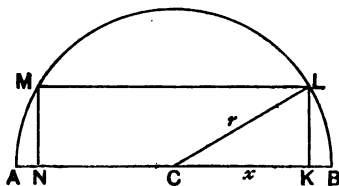


FIG. 24.

State and prove a relation that holds between KA , KL and KB , whatever values r and x have.

85. Draw to a scale of 10 feet to the inch the junction lines of a small tramway for a factory from the dimensions given in the sketch (Fig. 25). All lengths are in feet. The arcs AB and BC

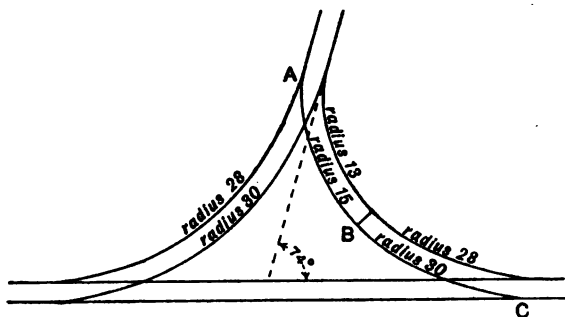


FIG. 25.

have a common tangent at B . What is the distance between the centres of the arcs AB and BC ? How does the knowledge of this distance enable you to find the centre of the arc BC , and the position of the point B ?

86. In the rough sketch in Fig. 26 the circle represents a section through the centre of the earth by a plane in which a star S lies. ab and cd are the directions in which the star S is seen from two places a and c , the star being so far away that ab and cd are parallel. ah and cf are tangents at a and c . Prove that the difference between the angles dcf and bah is equal to the angle aOc .

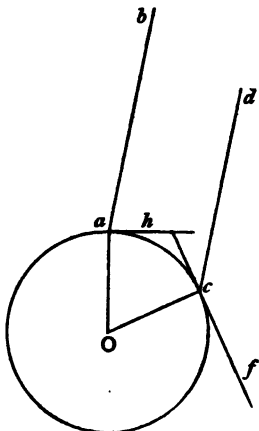


FIG. 26.

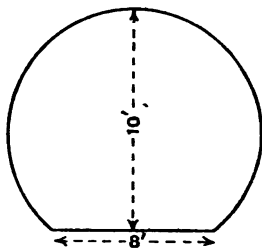


FIG. 27.

87. The section of a tube railway is a segment of a circle (Fig. 27). The breadth of the tube on the level of the rails is 8 feet, and its greatest height above the rails is 10 feet. Draw the section on any convenient scale, and write down the greatest breadth of the tube in feet to the nearest tenth.

State how you drew the section, and state and prove a property of the circle that you used.

Pythagorean Proposition.

88. Prove that of all straight lines drawn from a given point to a straight line the perpendicular is the least.

A instructs B to make a triangle with these measurements: two sides are to be 7 inches and 4.8 inches in length; also the angle opposite the shorter side is to be 45° . B says it is impossible to do so. Prove that B is right.

89. Given the lengths of two sides of a triangle, and of the perpendicular from the point where they meet on the third side, show how to find the length of the third side by drawing.

How many solutions are there? Find an expression for the length of the third side in terms of the given lengths.

90. A piece of wire ABC is bent at right angles at B , the length of AB being 4 cm. and of BC 12 cm. A point K travels from C to B , so that as its distance from C increases, its distance from A diminishes. Take K in a number of positions, measure KA for each, and show in a graph how KA varies as the distance KC increases. Find from your graph the length of KB when $KA = KC$.

A point K is to be found in BC so that KC may be bent into the position KA and so form a right-angled triangle ABK . Suppose BK to be x cm. long, and express the length KC in terms of x . Write down the relation between the three sides of the triangle ABK , and find a value of x that satisfies this relation.

Also find the position of K by a geometrical construction, justify the construction, measure BK , and tabulate the three results.

91. OX and OY are two lines at right angles, and a 60° set-square ABC ($\angle B = 60^\circ$, $\angle C = 90^\circ$) moves so that A and B travel along the lines OX and OY . Draw the path of C , using a length of 10 cm. for the hypotenuse AB .

What kind of line is the path of C ? Justify your statement.

92. BAC is a triangle, right-angled at A . Show that the square on BC is equal to the sum of the squares on BA and AC .

$ABCD$ is a quadrilateral figure in which $AB = 15$ cm., $BC = 25$ cm., $CD = 13$ cm., $AD = 11$ cm., and $AC = 20$ cm. Draw AE perpendicular to BC , and take P in EC 13 cm. from A . Show that BAC is a right angle, that BA is a tangent to the circle about ACE ; calculate BE , AE , PC , and show that AD is parallel to BC . The figure need not be drawn to scale.

93. A triangular piece of cardboard, ABC , is held over a horizontal table, the points a b c on the table being immediately below A B C . If a b c are corners of an equilateral triangle of sides 6 inches and their distances from A B C are 10 4 7 inches respectively, prove that ABC is an isosceles triangle, and calculate the lengths of its sides to the nearest tenth of an inch.

Maxima.

94. On a certain steamer the expense of coaling per mile is found to be directly proportional to the square of the speed maintained, being 2s. 6d. per mile when the speed is 30 miles per hour; the other expenses being £1. 16s. 0d. per hour at all speeds. Express in pounds the expense of a journey of 100 miles at x miles an hour; draw a diagram showing the expense of a journey of

100 miles at any speed from 0 to 45 miles per hour; and determine the most economical speed and the expense at that speed.

95. The shortest side of a triangle is 10 cm. long, the other two $10r$ cm. and $10r^2$ cm. long. Determine the greatest value r can have for a triangle to be possible. Obtain a first approximation (to within a tenth) by trial, then calculate the value to two decimal places, and draw the three lengths 10 cm., $10r$ cm., $10r^2$ cm.

96. ABC is a right-angled triangle having $AB = 8$ cm., $BC = 4$ cm., $\angle B = 90^\circ$. O is a point on AC , OL and OM $\perp$ s on BA and BC . Show by a graph how the area of the rectangle $OLBM$ changes as the perpendicular LO passes from A to BC , giving AL the values 0, 1, 2, 3, ... cm.

Compare the maximum area of the rectangle with the area of the triangle; compare also the rates at which O and L move along their respective lines.

97. Assuming that, if a gun is fired over a plain, the range of the projectile is given in yards by $y = 5000 \sin 2x$ (where x° is the angle of elevation of the barrel), plot out a curve showing the ranges between the limits of 5° and 20° for the angle of elevation. State what must be the value of x to give the greatest range.

98. A man in a launch at a point L is 30 miles from the nearest point A of a straight coast-line and wants to reach a point B on the coast 60 miles from A . He can either go by water the whole way or make for any point on the coast between A and B , where a motor car is running and watching to pick him up and finish the journey by land. Fill up a table like that below, and draw a graph to show how his total time from L to B varies according to the point of the coast at which he lands. And from your graph find what point he ought to strike in order to execute the whole journey in the least possible time. The speed of the launch is 10 miles an hour and of the motor 20 miles an hour. Use one division of your squared paper for a mile and 20 divisions for an hour; the time 0 hours need not be shown.

Distance from A of landing point in miles	0	10	20	30	40	50	60
Total time in hours							

Calculation of Triangles.

99. AB is the base (100 feet long) of a triangle ABC , and C is an inaccessible point. The angles ABC and BAC are found to be $63^\circ 14'$ and $80^\circ 10'$. Calculate the distance of C from A .

100. To determine the size of a piece of ground, pegs were driven at the four corners $A B C D$. AB was found to be 11·35 chains, AC 14·70 chains, AD 12·65 chains, $\angle BAC$ $26^{\circ} 40'$, $\angle CAD$ $43^{\circ} 0'$. Find the area of the ground by calculation.

101. A section of corrugated iron roofing consists of circular arcs of angle 110° , alternately curved in opposite directions. Supposing the section of the sheet to contain an exact number of these arcs, find (expressed as a decimal) the ratio of the area covered by it to the entire superficial area of the sheet-iron of which it is made.

102. A meteor is seen from two places, A and B ; B being 20 miles from A in a direction 15° west of south. At A the meteor appears to be $5^{\circ} 35'$ east of south, and at B it appears $7^{\circ} 24'$ north of east. The altitude of the meteor as observed from B is 23° . From these data find the height of the meteor.

103. Taking the earth as a sphere of radius 4,000 miles, find the distance from its axis of London whose latitude is $51^{\circ} 30' N$. Find also the distance London travels in an hour in consequence of the rotation of the earth.

104. The extreme range of the guns of a fort is 8,000 metres. A ship 14,000 metres distant sailing due east at 24 kilometres an hour notices the bearings of the fort to be $20^{\circ} 30'$ north of east. Find, to the nearest minute, when the ship will first come within range of the guns. Check your result by a drawing to scale.

105. Draw a triangle ABC such that $\tan A = 2$ and $\tan B = 0\cdot5$. Find, by measurement, the height h (perpendicular to AB) and the area S of the triangle you have drawn. Find the value of the ratio S/h^2 as a decimal.

Find an exact formula for the area of any triangle in terms of its height h and the tangents of its base angles A and B , and compare the numerical value of S/h^2 given by your formula with your previous result.

106. If one angle of a triangle is greater than another, prove that the side opposite the former angle is greater than the side opposite the latter.

State, with accompanying proof, whether the following theorem is or is not true:—'In every triangle, the magnitudes of the angles are proportional to the magnitudes of the opposite sides.'

107. A rod BC of length 5·8 cm. rotates about B . Another rod CA of length 8·6 cm. has one end C hinged to the first rod, while the other end A slides along a fixed line BO . By drawing the rods in various positions (taking values of $\angle B$ at intervals of 30° or so), find how the length BA varies as the angle B increases; and show

BA as a function of $\angle B$ in a graph for one revolution of BC , showing the actual length of BA and representing 30° by 1 cm.

Write down an equation connecting the angle B and the lengths of the three sides of the triangle ABC .

Solve the equation to find the length of BA when $\angle B = 35^\circ$. To check your result find the length by drawing.

108. Draw a circle of radius $R = 10$ cm. and draw two radii at an angle of $\theta = 80^\circ$. Join the ends of the radii, cutting off a segment of height h cm. and base c cm. Find a formula for the area of the segment in terms of R and θ . The area is sometimes taken as that of a triangle on the same base as the segment and of $4/3$ times the altitude of the segment. The formula $\frac{2ch}{3} + \frac{h^3}{2c}$ is also used to give the area. Use these three methods to determine the area of the given segment, and tabulate your three results.

109. Fig. 28 represents a two-foot rule, which is pivoted at O and hinged about ac and bd . Imagine it placed, as shown, flat on a table, so that the angle AOB is 34° , and calculate the distance from A to B .

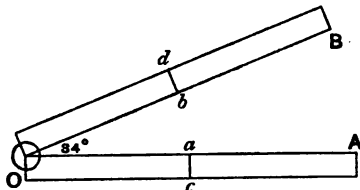


FIG. 28.

If the parts Ac and Bd are now turned about the hinges ac and bd through 90° from the plane of the table, what is the new distance from A to B ? Briefly give reasons for your statements.

110. Sufficient data are given below to draw three triangles. State, with reasons, whether any two of the triangles are similar to one another:

- One angle 90° , hypotenuse 15 cm., one side 9 cm.
- Two sides 3.6 in. and 4.8 in., included angle 60° .
- Sides 36 yards, 27 yards, 45 yards.

Equations.

111. The speed (s) of sound in air at a temperature of t° (Fahrenheit) is given by the expression $s = a + bt$. When $t = 32$, $s = 1,087$ ft. per sec.; and when $t = 41$, $s = 1,096$ ft. per sec. Find the value of s when $t = 70$.

112. A man pays an annual subscription of £10 to a fishing club, and each day he goes fishing he spends 10s. in addition. Find an equation connecting y the total cost in pounds of a day's fishing, including the proportion of the subscription, with x the number of days he goes fishing during the year. Draw the graph of the equation supposing x to range from 5 to 50.

113. A retail bookseller is supplied by a wholesale firm, who allow discount off the published price of certain books (class A) at 40 per cent. ; of all others (class B) at $33\frac{1}{3}$ per cent. At the end of a year he finds that he would have paid $2\frac{1}{4}$ per cent. less than he actually did, had he accepted the tender of another wholesale firm, who offered a uniform discount of $37\frac{1}{4}$ per cent. What percentage of his purchases (reckoned at published price) belonged to each class? Give your answer to the nearest integer.

114. The diameter d feet of a pipe connecting a certain tank to a water crane is given by the following equation :

$$d^4 = \frac{1}{1000} \left(2 + \frac{9}{d} \right).$$

Solve this equation graphically.

115. Given $x^2 - 3xy + y^2 = 0$, and $ax = by = 1$, find an equation which contains only a , b , and numbers, and deduce the value of a/b .

116. From the relation $v = u + at$ express t in terms of v u a ; then substitute this value in $d = ut + \frac{1}{2}at^2$ and so obtain the relation $v^2 = u^2 + 2ad$. Find the values of t and d when $v = 100$, $u = 84$, $a = 32$.

117. It is believed that the cost of keeping a school varies according to the law

$$y = ax^2 + bx + c,$$

where 50y pounds sterling is the cost, and 50x is the number of boys. It was found that with 100 boys, the cost was £2,050; with 150, £2,600; with 200, £2,950. Find a b c , and deduce the cost for 170 boys.

Verify by a graphical method.

118. It is known that when a certain tackle is used to raise heavy weights, the weight W lb. that can be raised by a force of P lb. is given by the formula

$$W = kP - 70,$$

where k is some number. And a man finds that by pulling with a force of 100 lb. he can raise a weight of 330 lb. Find what the number k is.

Find, further, what weight a force of 140 lb. will raise, and what force is needed to raise a weight of 400 lb.

119. A and B are two coal pits m miles apart. The price of coal at A is a shillings a ton, and at B it is b shillings a ton. The cost of carriage from each pit is c shillings per ton per mile. Give, in terms of a , b , c , the distance from A of a point X , on the road from A to B , at which the total cost of the coal is the same, whether it comes from A or from B . Give a numerical result when $a = 29$, $b = 27$, $c = 1.5$, $m = 20$.

120. If $\frac{2}{x^2 - 3x} + \frac{5}{x^2 + 3x} + \frac{A}{x^2 - 9} = \frac{2x^2 + 6x - 9}{x^3 - 9x}$ for all values of x , find the value of A in terms of x .

121. A screen is placed between two lights A and B , whose lighting powers are as 3 to 4. The intensity of illumination of the screen by either light varies inversely as the square of the distance of the screen from the light. A and B being 5 feet apart, find the distance, x feet, from A at which the screen is equally illuminated by the two lights. Verify your result, and state what interpretation you place on the negative value for x .

122. Solve $a^2 = b^2 + c^2 - 2bc \cos A$ as a quadratic equation in which b is the unknown quantity. And hence, or otherwise, calculate the positive value of b when $a = 11$ cm., $c = 9$ cm., $A = 40^\circ$. Check your result by an accurate drawing.

123. The formula for bodies projected vertically upwards is $s = ut - \frac{1}{2}gt^2$, where t = time in seconds, s = height in feet above starting point at time t , u = initial velocity in feet per second, and $g = 32$.

Find the time a body takes to rise 192 feet vertically when the initial velocity is 128 feet per second.

Indices.

124. Find the values, giving not more than three significant figures, assumed by $a + a^4$, when a has the successive values 200, 20, 3, 2, 1, 0.2, 0.02.

Tabulate the values of a , a^4 , $a + a^4$, and explain what your table shows as to the relative importance of the two terms according as a is large or small.

125. When a man lends £ a at b per cent. compound interest, the value of his loan grows so that after c years it is £ $a \left(1 + \frac{b}{100}\right)^c$. Use this formula to find the value after two years of a loan of £460 at $3\frac{1}{2}$ per cent. Give the value to the nearest shilling.

126. The formula $\frac{nr^{n-1} - r^{-n}}{n-1}$ is used by engineers. Find its value to two decimal places when $n = 1.05$ and $r = 2$.

127. Find between what pair of consecutive integers each of the following lies: 1.1^4 , 1.1^8 , $2^{1/4}$, $3^{1/2}$.

128. The discharge Q of water through a rectangular submerged hole is given in cubic feet a second by the equation

$$Q = \frac{2}{3} CB (H^{3/2} - K^{3/2}) \sqrt{2g},$$

all the dimensions being in feet. Find what the discharge is in gallons per 24 hours, when $C = 0.60$, $g = 32$, $B = 8$, $H = 2$, $K = 1$. One cubic foot = 6.2 gallons.

129. The pressure p and volume v of a gas which expands in a vessel impervious to heat are connected by the relation $pv^\gamma = \text{constant}$, where $\gamma = 1.4$. A mass of gas occupies a volume of 1 cubic foot at a pressure of 100 lb. per square inch. Find its pressure when its volume is 2, 3, 4, . . . 10 cubic feet respectively, and represent the relation between pressure and volume within these limits by a curve drawn on squared paper.

130. After very careful calibration it was found that the discharge D gallons per minute over a certain weir was given by the equation

$$D = 20H^{1.48}$$

when H was the head of water in feet above the sill of the weir. Calculate with the aid of a table of logarithms the head necessary to discharge 3.72 gallons per minute.

131. A working rule for the collapsing pressure of a boiler flue is given by $P = \frac{806300t^{2.19}}{LD}$, where P = the collapsing pressure in lb. per sq. inch, D = diameter of the flue in inches, L = the length in feet, t = the thickness of the plate in inches. Find the collapsing pressure when $t = \frac{3}{8}$, $D = 8\frac{1}{4}$, $L = 25$.

Co-ordinates.

132. Draw OX horizontally towards the right and OY vertically upwards. Values of x are measured to the right along OX from O and values of y from O along OY , unit 1 foot. A telegraph wire is given in position approximately by the equation

$$(100 - x)^2 = 1000(y - 50).$$

The centre line of the jib of a builder's crane is given in its highest position by the equation $y = 3 + \frac{1}{2}x$. The height of the lower end of the jib above the ground (that is, above OX) is 30 ft. and the length of the jib is 43 ft. Find whether the crane fouls the wire.

133. Draw the quadrilateral $ABCD$, the corners A, B, C, D having the co-ordinates (14, 12), (12, 1), (3, 10), (0.4, 14.6).

How many conditions does it take to fix a triangle in size and shape? Hence discuss how many conditions will fix a quadrilateral. How many conditions are used above to specify $ABCD$ by co-ordinates? Can you reconcile these statements?

Discuss how many conditions will fix a polygon of N sides (1) in shape and size, (2) in position, and give two ways of specifying such a figure in each way.

Could you make a quadrilateral frame with its sides equal to those of $ABCD$? Could you make a frame satisfying these conditions, and also having each diagonal 15 cm. long? Justify your answers.

Suppose $ABCD$ to be a frame made of four rods jointed together, and suppose AB fixed. Draw the path that O , the middle point of CD , is able to trace out. Calculate the co-ordinates of O in the position given in the first paragraph, and also when O is as far as possible from A , and check your results by measurements of your drawing.

Solid Geometry.

134. Assuming that sand will just rest with its surface inclined to the horizon at an angle of 30° , find the smallest possible diameter of the base of a conical heap, containing 40 cubic feet of sand.

135. A square house, measuring 30 feet each way, has a roof sloping up from all four walls at 35° to the horizontal. Find the area of the roof.

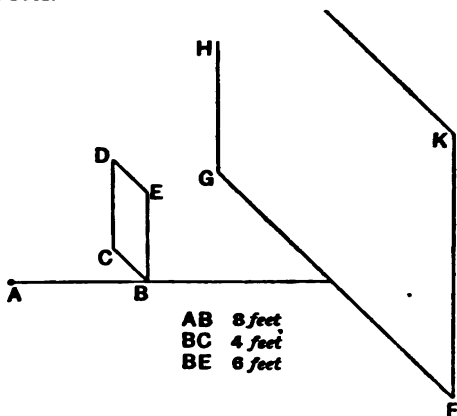


FIG. 29.

136. A light, which may be shown as a point, is placed on a horizontal floor at A (Fig. 29), and shines through a vertical

rectangular opening $BCDE$ on to a vertical screen $FGHK$ parallel to the opening and 12 feet from it. The angle ABC is a right angle. Draw the shape, and calculate the dimensions, of the portion of the screen which is lighted up, using the additional data given below the figure.

Examine the effect on the bright portion of moving the light 2 feet horizontally at right angles to AB .

137. In the map of a mountain district in Bavaria (see Fig. 30)

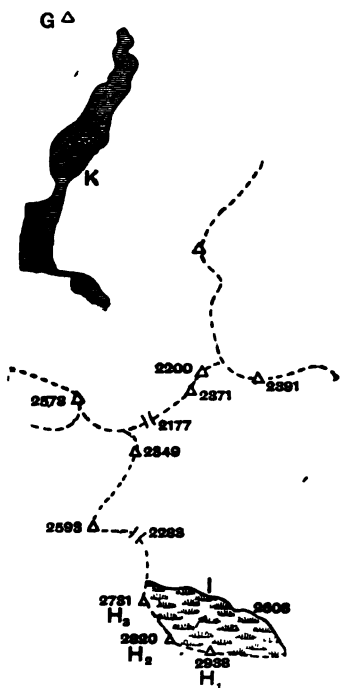


FIG. 30.

K is a lake, I a level ice field backed by mountains H_1 H_2 H_3 , the dotted lines represent mountain ridges, Δ indicates summits, X the lowest points on the ridges, and the numbers indicate altitudes in metres. Examine whether any portion of the range $H_1H_2H_3$ is visible from the lake. How many metres must a person ascend from the lake in the neighbourhood of G in order (1) to just see the tip of H_1 , (2) to see the three summits, H_1 H_2 H_3 ?

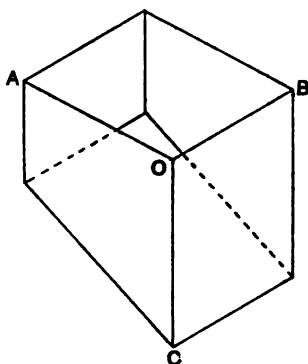


FIG. 31.

138. Fig. 31 is a projection of an open tank with a sloping bottom, so drawn that the edges OA , OB , OC and all lines on the tank parallel to them are 50 times as long as they are made in the figure. The tank is made of sheet-iron. Find the area of sheet-iron used, and how many gallons the tank will hold.

MAIR

B b

Find expressions for the number of gallons in the tank for any depth of water at the deep end, including the case when the bottom is not completely covered. Draw a graph showing the volume of water for any depth.

Use this graph to find the depth of water when the tank contains 40 gallons, and when it contains 200 gallons. Find these depths, also, by writing down and solving suitable equations.

The edges OA , OB , OC being all foreshortened to the same extent by the projection, find the amount of foreshortening. Each line OA , OB , OC on the projection being $1/50$ of the actual length of the edge, find the scale of the projection.

Supposing the water to flow in at the rate of a gallon a second, find from your graph, or by calculation, the rate at which the depth increases when there are in the tank, 0, 20, 40 . . . gallons, and draw a graph giving the rate of filling in terms of the quantity already in the tank.

139. What do you consider is, in Geometry (a) the purpose of a definition, (b) the test of a good definition?

Read carefully through each of the following defective definitions in turn; then, briefly and relevantly, referring to each by its number, point out wherein the defect lies.

(i) A circle is the assemblage of all those points which are equally distant from a fixed point, called the centre.

(ii) Two straight lines are called parallel when, however far produced, they never meet.

(iii) A circle is a plane figure closed by a boundary such that all straight lines which are terminated by the boundary and pass through a certain point within called the centre are bisected at that point.

(iv) A plane is a surface on which an unlimited number of straight lines can be drawn.

TABLE OF WEIGHTS AND MEASURES

1 metre = 100 cm. = 1,000 mm. = 0·001 km.

1 yard = 3 ft. = 36 in. = 0·9144 metre.

1 mile = 8 fur. = 80 chains = 320 po. = 1,760 yds.

1 hectare = 100 ares = 10,000 sq. metres.

1 acre = 4 roods = 10 sq. chains = 4,840 sq. yds. = 0·4047
hectare.

1 litre = 1 c. dm. = 100 cl. = 0·01 hl. = 0·001 c. metre.

1 gallon = 4 qt. = 8 pt. = 0·1606 c. ft. = 4·546 litres.

1 quarter = 8 bush. = 32 pk. = 64 gal.

1 kilogram = 1,000 g. = 0·001 tonne.

1 lb. = 16 oz. = 7,000 grains = 453·6 grammes.

1 ton = 20 cwt. = 80 qr. = 2,240 lb.

1 litre of water weighs 1 kg.

1 c. ft. of water weighs 62·3 lb.

Ratio of circumference of circle to diameter is 3·1416.

ANSWERS TO CERTAIN NUMERICAL QUESTIONS

CHAPTER III

2. 6 or 7 hectares or 16 acres. 4. 200 sq. cm.
 6. 1,010,000; 994,000. 8. Areas $10\cdot8$ and $11\cdot88$ sq. in.
 10. 34,000 sq. yds. or $7\cdot1$ acres. 11. $11\cdot56$, $13\cdot6x$, $0\cdot5$ sq. in.
 12. 11 layers. 13. 470 sq. ft. 14. 7 mm./sec.
 15. $2\cdot6$ sq. in.; 18 or 19 ft. 17. $0\cdot27$ acre. 18. $2\cdot9$ cm.
 21. 11 min. after first man's start, distance 400 yds.

CHAPTER IV

5. $68\cdot9$ in. 9. $0\cdot7$ kg.; $1000bc/a-d$.
 11. £18. 8s. 4d.; $\pounds pqr/240$. 12. 180 sq. m.; 1,188 c. m.
 13. 302 and $26880/mn$ sq. ft. 14. 18 pence per ton.
 15. 9s. 8d.; $0\cdot000,009,13 mnba$ shillings.
 16. $bclx/2240$ or $2\cdot25$ tons; k/bc or 28 ft. per min.
 17. 8,800 sq. ft.; 180 people. 18. $0\cdot13a$ in.
 19. 3,000,000 c. ft. or 20,000,000 gal. 20. 36%.
 21. 9,000 gal./min.

CHAPTER V

4. 7 miles. 9. $4\cdot6$ cm.; 40 sq. cm.; 59 sq. cm.
 14. 4 ac. 1 rd. 15. 73 gals./min.

CHAPTER VI

1. $0\cdot8$ lb. 2. 3 lb. 3. 482; $7\cdot2$ lb.
 4. $106\cdot6$ c. c. 6. 20 H.P., to nearest integer.
 7. Average price $\pounds(ax+by+cz)/(a+b+c)$ or £34.
 11. $0\cdot40$ cm. 12. 57 and 77%. 13. $0\cdot615$ in.
 14. 87 yds.; $3y = 2x + 20$. 15. $2\cdot1$, $0\cdot97aa$, $8\cdot7aa$ sq. ft.
 16. £920. 20. 3.20 p.m.

CHAPTER VII

1. 6.4 m. 4. 3.6 m.; raised 0.9 m.
 7. Limits 3.8 m. in front of O and 1.2 m. behind.
 8. 10.1 ft. from floor, 4.1 ft. from line. 11. 204 m.
 12. $AP = 4.95$ cm., $AO = 7.0$ cm., $\angle OAP = 45^\circ$, $\angle OAB = 60^\circ$.
 13. $CB = 1,580$ m., $CD = 830$ m.

CHAPTER VIII

7. Area 5.1 sq. cm. 10. $C = E = 150^\circ$, $D = 60^\circ$.
 11. $K = 120^\circ$, $L = 90^\circ$, $M = 45^\circ$, $N = 105^\circ$; K and M increase by 10° , L and N diminish by 10° . 17. $10\frac{1}{2}$ acres.
 18. $h(a/2 + b + c + d + e/2)$ or 75 sq. ft. 19. 15.8 lb.
 21. 108 hours. 23. $(c-b)/(a-b)$ or $2/3$.

CHAPTER IX

13. Limits for b/c are 0.42, 0.50, 0.64, 0.71.
 15. 12° , 24° , 37° , 53° . 17. 0.309.
 18. $df = 3.1$ sq. cm., $DF = 37$ sq. cm., $de/DE = 0.29$, $df/DF = 0.084$. 19. 12 ft.; 37 or 38 stones. 21. 4.7 min. later.
 22. 40 tons. 24. 0.016 in.; 0.00020 sq. in.
 25. $y/(x-0.9) = 0.7$, or other forms.

CHAPTER X

1. 256, n^2 . 2. 1,122. 3. 7,482.
 4. 507 red, 507 white, 527 black. 5. 28. 8. 4 cwt.
 9. 0.72 gram. 10. 33 cwt.; 1%. 11. 16; 55.875, say 56.
 12. 1%. 13. £618. 14. $\frac{1}{2}ax_1^2 + bx_1$; $y = px + c$.
 15. Sines 0.39, 0.96; angles 12° , 25° , 57° .
 18. 11 million gallons.

CHAPTER XI

1. $n = 68^\circ$, $CW = 52$, $TW = 25$, $CZ = 147$, $TZ = 69$; $OE = 49$, $CA = 56$, $TF = 23$, $TB = 26$. 2. 24° , 0.92 , 0.44 .
 3. 53° , 0.80 , 1.33 . 4. 76° , 0.97 , 0.24 . 9. 0.69 , 0.97 .
 10. 3.7 in. 11. 27° . 12. 185 and 249 yds.
 13. D is 4.04 ch. north and 8.88 ch. east.
 14. Each ratio = 1.62. 16. 1.13 in.; 16° .
 17. $(6-x)/a + \sqrt{86+x^2}/b$ hours; $x = 3.5$. 18. 14.1 in.
 19. 3.53, 7.28, 11.55 in.; greatest swing 86.9° . 20. 12.
 21. 39%. 22. 1,944 cases; 36s. 10d. 23. About 2,380.
 24. 4.7, 0.6.

374 ANSWERS TO NUMERICAL QUESTIONS

CHAPTER XII

Art. 8. For indices 20, 40, ... 240, the powers of 1.01 are 1.220, 1.489, 1.817, 2.217, 2.705, 3.300, 4.027, 4.914, 5.996, 7.316, 8.927, 10.893.

- Ex. 2.** At C at 8.40 a.m. 3. 1,600,000 gal.; 290 gal.
 4. 7; 8 more. 5. £86. 6. 0.84.
 7. 99 gal.; true volume 103 gal. 8. 32,000 miles.

CHAPTER XIII

21. 9,200 c. ft. 24. 5.12 kilos; 32 shillings.
 26. 750 gal./sec. 27. 24,000 tons. 28. 6 ft. 9 in.
 31. 8,000 miles. 32. 13 ft.

CHAPTER XIV

1. 8.6 cm. 2. 23.1. 4. 400 ft.; 2.5 sec.
 6. 200 lb./sq. in. 7. 10.2, 12.1, 15.2 cm.
 8. In millions (i) 38.93, 40.05; (ii) 38.89, 40.01.
 9. 65 million miles. 11. 6.9 m.
 12. Son's weight 73% of father's, but he needs 81% of father's food, i.e. 5.7 oz. 13. $bc/2$ or 14.9 sq. in. 14. 43 burners.
 15. 0.54; 1.41 in. 16. 48 swings. 17. 7,100 ft.
 18. 2.9%. 19. 0.141. 21. 13.1 lb./sq. in.
 23. 63 ft. away horizontally. 24. $x = 1.22$, $k = 470$, $p = 49$.
 25. 1.61, 5.35. 27. 210 kilos. 32. 100.8 cm.

CHAPTER XV

11. 86 ft. 13. Side of table 15.3 in.
 17. $m = 3.08$ gives 1.66 for least value of D .
 19. sine = 0.419, cosine = 0.908. 20. 11.3 cm.
 21. 30 in. 22. 103,000 gal.; 1.9 ml./hr.
 24. 71,000 miles. 26. 9 ft.

CHAPTER XVI

Art. 16. Area 40.0 sq. cm.

Art. 17. (1) $A = 26.4^\circ$. (2) $a = 11.1$. (3) $c = 7.61$ or 5.49 .

Art. 18. $a = 12.5$, $b = 8.41$, $A = 108.9^\circ$, $B = 39.6^\circ$, $C = 31.5^\circ$.

Ex. 5. 68 yds. 6. 3.1 miles. 7. 2.8, 0.7.

8. 19s. 2d. 9. $a = 0.39$, $b = 0.174$.
 10. 7.4 , 10.5 , 15.1 ; $y = 2.75 + 1.55x$. 14. 0.5, -0.6.
 15. 117° , 0.89, -0.45. 17. 5.36 km.; 444 m.; 4.8° .
 18. 980 ft. 19. 376 metres.

20. $x = 8.13$, $y = 6.45$. 21. $y = 1.62x - 5.3$.
 22. 80 ft. at 1 sec. and 5 secs., 128 at 2 and 4, 144 at 3 secs.
 23. 4.24, 4.07, 4.01 inches. 24. 205° Fahr.
 25. $\frac{1}{2}c^2/\sin A \sin B/\sin(A+B)$; 308 c. yds. 26. 80 lb./yd.
 27. 14.9 or 15.0 cm. 28. 760 ft. 29. 280 sq. ft.
 31. 117 ft. 32. 219 links 71.7° S. of E.
 33. 153.3° , 5,090 links.

CHAPTER XVII

1. 6.9 cm. 9. 0.00008. 10. 0.00016. 13. 258° .
 Art. 52. $AB = 2.2$, $BC = 2.94$, $CA = 3.09$, $AL = 2.51$, $BL = 2.42$,
 $CL = 1.78$.
 15. 13,000 c. ft. 17. 980 ft. 20. 3.69 in.; 21.6° .
 21. 5.0 ft. 22. 8,748 c. ft. 24. 21 in 100.
 25. 896 ft. 26. 253 ft. 28. 120° .
 30. 3 hrs. 54 min. 31. 1,050 c. ft./min.; 4.9 ml./hr.
 32. In links to nearest integer $AB = 800$, $CD = 118$.

INDEX TO EXPLANATIONS OF TERMS AND PROPERTIES OF FIGURES

References are to pages.

- Acute $\angle$, 33 footnote, 71.
- Acute angled Δ , 33 footnote.
- Alternate $\angle$ s, 129.
- Alternate segment of a circle, 225.
- Angle, 3, 4.
- Acute $\angle$, 33 footnote, 71.
- Alternate $\angle$ s, 129.
- $\angle$ between two planes, 306.
- $\angle$ greater than 180° , 219.
- Bisection of an $\angle$, 22, 64.
- Corresponding $\angle$ s, 129.
- Degree of $\angle$, 4.
- Minute of $\angle$, 69 footnote.
- Obtuse $\angle$, 33 footnote, 71.
- Right $\angle$, 3.
- Supplementary $\angle$ s, 346 footnote.
- To make an $\angle =$ a given $\angle$, 64.
- Vertically opposite $\angle$ s, 24.
- Angle-sum property of Δ , 127.
- Approximation, suitable degree of, 29, 47.
- Arc of a circle, 220.
- Areas, 27.
- Circle, 51, 323.
- Parallelogram, 188.
- Rectangle, 29.
- Similar figures, 166.
- Surface of sphere, 332.
- Triangle, 33, 291.
- $(a \pm c)^2 = a^2 + c^2 \pm 2ac$, 257.
- Average, 51.
- Axes of co-ordinates, 116.
- Base of logarithm, 200.
- Base of Δ , 35.
- Binomial series and theorem, 181.
- Bisection of an $\angle$, 22, 64.
- Bisection of a straight line, 13.
- Brackets, use of, 23, 56.
- Centre of circle, 1.
- To find centre of circle, 216.
- Centre of sphere, 100.
- Chord of circle, 5.
- Circle, 1, 99, 303.
- $\angle$ in given segment is constant and half $\angle$ at centre, 220.
- Arc of a circle, 220.
- Area, 51, 323.
- Centre, 1.
- Chord, 5.
- Chord properties, 6, 7, 8, 9.
- Circumference, 49.
- Construction from various data, 225, 226.
- Contact of circles, 8.
- Contact of circle and line, 72.
- Number of points that fix a circle, 216.
- Radius, 1.
- Ratio of circumference to diameter, 50, 323.
- Rectangle properties of circle, 227.
- Sector of circle, 331.
- Segment of circle, 220.
- Tangent to circle, 72, 222.
- To find centre, 216.
- Circumference of a circle, 49.
- Coefficient, 179.

- Combinations of n things p at a time, 178.
- Commensurability, 160.
- Compass, points of the, 8, 11.
- Concurrent, 217.
- Concyclic, 238.
- Cone, 381.
- Congruence, 78.
 - Δ s with same $a b c$ congruent, 66.
 - Δ s with same $a b C$ congruent, 67.
 - Δ s with same $a B C$ congruent, 75.
 - Δ s with same $a A O$ congruent, 76.
 - Discussion of Δ s with same $a b B$, 70-75.
- Congruent solids, 321.
- Constraint, degrees of freedom and, 102, 105, 818.
- Contact of circles, 8.
- Contact of circle and line, 72.
- Contractions and symbols, their right use, 28, 30, 40, 55.
- Converse propositions, 14.
- Co-ordinates, 116.
 - Origin and axes, 116.
 - Plane co-ordinates, 117.
- Corresponding L s, 129.
- Cosine of an L , 194.
- Cosine of an obtuse L , 262.
- Counterclockwise, 219.
- Cylinder, cylindrical, 58, 110, 329.
 - Axis of cylinder, 329.
 - Volume of cylinder, 48.
- Danger angle, 216.
- Degree of angle, 4.
- Degrees of freedom and constraint, 102, 105, 818.
- Diagonal, 81, 135.
- Elevation and plan, 335.
- Ellipse, 119, 120.
- Ellipsoid, 119.
- Equation, root of equation, solution of equation, 84.
 - Independence of equations, 269, 274, 278.
- Quadratic equations, 92, 264, 291.
- Simple equations, 86.
- Exterior L of Δ , 126.
- Freedom and constraint, degrees of, 102, 105, 818.
- Function, 36, 55.
- Graph, 85.
- Height of Δ , 85.
- Hexagon, 42.
- Horizontal plane, 107.
- Hypotenuse of right-angled Δ , 41.
- Image in a line, 8.
- Image in a plane, 306, 321, 323.
- Independence of conditions, 269, 274, 278.
- Index notation, 152.
 - Fractional index, 242.
 - Laws of indices, 209, 210.
 - Negative index, 211, 241.
 - Zero index, 211.
- Interior and opposite L of a Δ , 126.
- Isosceles Δ , 6.
- Isosceles Δ has base L s equal; converse; 6, 7.
- Locus, 10.
- Logarithm, 200, 236.
- Mean proportional, 231.
- Minute of L , 69 footnote.
- Negative quantities, 87, 116.
- Obtuse L , 38 footnote, 71.
- Obtuse angled Δ , 38 footnote.
- Origin of co-ordinates, 116.
- Parallels, 18, 105, 128.
 - Equality of corresponding L s, of alternate L s; converses; 129, 130.
 - Parallel planes, 107, 306.
 - L s to a plane are $\parallel$; converse; 304.
 - Transversals are cut by $\parallel$ s in same ratio, 158.

- Parallel rulers, 182.
 Parallelepiped, 844.
 Parallelogram, 183.
 Area, 188.
 Opposite sides equal; converses; 185.
 Pentagon; regular pentagon; 838 footnote.
 Perimeter, 270, 827.
 Perpendicular, 5.
 $\perp$ is shortest distance to a line, 189.
 $\perp$ to a plane, 106, 299.
 $\perp$ s to a plane are $\parallel$; converse; 304.
 Property of $\perp$ bisector of a line, 11.
 To draw a $\perp$ to a line, 18, 14, 221.
 Plan and elevation, 885.
 Plane, 100.
 Plane table, 62.
 Plumb line, 107.
 Polygon, 140, 827, 838 footnote.
 Perimeter, 827.
 Regular polygon, 838 footnote.
 Polyhedron, 838.
 Power, 152.
 Prism, 820.
 Projection, 108, 115.
 Projection on a line, 287.
 Projection on a plane, 334.
 Plan and elevation, 835.
 Proportion, 281.
 Proportional, mean, 281.
 Pyramid, 299.
 Volume, 811, 820, 836.
 Pythagoras, theorem of; converse; 259.
 Extension to any Δ , 290.
 Quadrant, a quarter of a circle bounded by two $\perp$ radii.
 Quadratic equations, 92, 264, 291.
 Quadrilateral, a figure of four sides.
 Sum of opposite $\angle$ s of cyclic quadrilateral is 180° , 222.
 Radius of circle, 1.
 Ratio, 150.
 Ratio of circumference of circle to diameter, 50, 828.
 Rectangle; rectangular; 27, 28, 186.
 Area of rectangle, 29.
 $(a \pm c)^2 = a^2 + c^2 \pm 2ac$, 257.
 Regular pentagon and polygon, 838 footnote.
 Right $\angle$, 8, 4.
 Right Δ , 81.
 Calculation of, 268.
 Hypotenuse of, 41.
 To draw, from various data, 260.
 Theorem of Pythagoras; converse; 259.
 Root of equation, 84.
 Sector of circle, 881.
 Segment of circle, 220.
 Set-square, 181.
 Shape of Δ , conditions that fix it, 168.
 Similar figures, 161, 167.
 Similar solids, 825.
 Similarity of Δ s, conditions that ensure it, 164.
 Simple equations, 86.
 Sine of an $\angle$, 165.
 Sine of an obtuse $\angle$, 232.
 Slide rule, 209.
 Slider crank pair, 78.
 Solid body, position fixed by six data, 812.
 Congruent solids, 821.
 Solution of equation, 84.
 Sphere, 100.
 Area, 832.
 Centre, 100.
 Section by a plane, 829.
 Volume, 833.
 Square, 86, 186.
 To draw square = a given rectangle or Δ , 229.
 Square root, 212, 244, 248.
 Straight path shortest, 188.
 Supplementary $\angle$ s, 846 footnote.
 Surface, 99.
 Symbols, table, 23.
 $>$ and $<$, 155.
 $\sqrt{\quad}$, 218.

- Precedence of $\times$ over $+$, 29.
- Right use of symbols and contractions, 28, 30, 40, 55.
- Table of weights and measures, 371.
- Tangent of an L , 195.
- Tangent of an obtuse L , 284.
- Tangent to circle, 72, 232.
 - L between tangent and chord = L in alternate segment, 225.
 - To draw a tangent, 280, 358.
- T-square, 181.
- Tetrahedron, 815.
- Transversal, 129.
- Trapezium, 189.
- Triangle, 8, 32.
 - Acute $Ld \Delta$, 88 footnote.
 - Angle-sum property, 127.
 - Area, 83, 291.
 - Base, 35.
 - Bisector of vertical L divides base in ratio of sides, 191.
 - Conditions that fix Δ , 65.
 - Congruence of Δ s, 65.
 - Exterior L of Δ , 126.
 - Greater L opposite greater side; converse; 190.
 - Height of Δ , 35.
 - Interior and opposite L , 126.
 - Isosceles Δ , 6.
 - Obtuse $Ld \Delta$, 88 footnote.
 - $\parallel$ to base cuts sides in same ratio, 157.
 - $\perp$ bisectors of sides are concurrent, 217.
 - L s of a Δ are concurrent, 289.
 - Pythagorean theorem; generalization; 259, 290.
 - Right $Ld \Delta$, 81.
 - Shape of Δ , conditions that fix, 163.
 - Similarity, 164.
 - Δ given by co-ordinates, 292.
 - Two sides are greater than third, 8.
 - Vertex, 123.
- Trisection of line, 146.
- Vertex of Δ , 123.
- Vertical line, 107.
- Vertical plane, 107.
- Vertically opposite L s, 24.
- Volumes, 45.
 - Cylinder, 48.
 - Rectangular room, 45.
 - Similar solids, 167, 386.
 - Sphere, 333.
 - Pyramid, 811, 817, 386.
- Weights and measures, table of, 371.

UNIV. OF MICHIGAN,

APR 14 1914

OXFORD

PRINTED AT THE CLARENDON PRESS

BY HORACE HART, M.A.

PRINTER TO THE UNIVERSITY

OXFORD UNIVERSITY PRESS

**PUBLICATIONS IN NATURAL SCIENCE
AND MATHEMATICS**

NATURAL SCIENCE

Botany

Series of Botanical Translations, under the general editorship of Professor I. BAYLEY BALFOUR

Royal 8vo, unless otherwise specified. The prices are for copies with morocco back and copies in cloth respectively.

Jost's Lectures on Plant Physiology. Authorized English translation by R. J. HARVEY GIBSON. 24s. net and 21s. net.

Schimper's Geography of Plants, authorized English translation by W. R. FISHER, revised by P. GROOM and I. BAYLEY BALFOUR. With maps, collotypes, portrait, and 497 other illustrations. Morocco back, £2 2s. net.

Pfeffer's Physiology of Plants, a treatise upon the Metabolism and Sources of Energy in Plants. Second fully revised Edition, translated and edited by A. J. EWART. Vol. I, £1 6s. net and £1 3s. net. Vol. II, 16s. net and 14s. net. Vol. III completing the work, £1 1s. net and 18s. net.

Goebel's Organography of Plants. Authorized English Edition by I. BAYLEY BALFOUR. PART I, General. 12s. net and 10s. net. PART II, Special. 24s. net and 21s. net.

Goebel's Outlines of Classification and Special Morphology of Plants. Translated by H. E. F. GARNSEY, and revised by I. BAYLEY BALFOUR. Morocco back, £2 2s. net (a few copies only remain).

Sachs's History of Botany (1580-1860). Translated by H. E. F. GARNSEY, revised by I. B. BALFOUR. 2nd impression. Cr. 8vo, cloth, 10s. net.

A History of Botany, 1860-1900: a continuation of Sachs's History. By J. REYNOLDS GREEN. Crown 8vo. 9s. 6d. net.

De Bary's Comparative Anatomy of the Vegetative Organs of the Phanerogams and Ferns. Translated by F. O. BOWER and D. H. SCOTT. £1 4s. net and £1 1s. net.

De Bary's Comparative Morphology and Biology of Fungi, Mycetozoa and Bacteria. Translated by H. E. F. GARNSEY, revised by I. BAYLEY BALFOUR. £1 4s. net and £1 1s. net.

De Bary's Lectures on Bacteria. Second edition. Translated by H. E. F. GARNSEY, revised by I. BAYLEY BALFOUR. Crown 8vo, cloth, 5s. net.

Solms-Laubach's Introduction to Fossil Botany. Translated by H. E. F. GARNSEY, revised by I. BAYLEY BALFOUR. 17s. net and 15s. net.

Fischer's Bacteria, translated by A. COFFEN JONES. Cloth, 7s. 6d. net.

Knuth's Handbook of Floral Pollination, based upon Hermann Müller's work, *The Fertilization of Flowers by Insects*, translated by J. R. AINSWORTH DAVIS. Vol. I, 21s. net and 18s. net. Vol. II, 35s. net and 31s. 6d. net. Vol. III, 31s. 6d. net and 28s. net.

Solereider's Anatomy of the Dicotyledons. Translated by L. A. BOODLE and F. E. FRITSCH. Revised by D. H. SCOTT. Two volumes, 27s. 6d. net each and 24s. net each.

Warming's Oecology of Plants. English edition by P. GROOM and I. B. BALFOUR. 10s. net and 8s. 6d. net.

On the Physics and Physiology of Protoplasmic Streaming in Plants. By A. J. EWART. With 17 illustrations. 8s. 6d. net.

List of British Plants. By G. C. DRUCE. Crown 8vo, paper covers. 2s. 6d. net; cloth interleaved, 3s. 6d. net.

CLARENDON PRESS BOOKS

Floral Mechanism : Diagrams and Descriptions of Common Flowers.

By A. H. CHURCH. Royal 4to. Part I, Types I-XII (Jan. to April). With coloured plates and numerous other figures. 21s. net.

Index Kewensis; an enumeration of the Genera and Species of Flowering

Plants from the time of Linnaeus to the year 1885. Edited by Sir J. D. HOOKER and B. D. JACKSON. 2 vols. 4to, morocco back, £10 10s. net.

Supplement I (1886-1895), can be ordered from Mr. Frowde, price with the Index £12 13s. net; it is not sold separately. Supplement II (1896-1900), 28s. net. Or Fasc. I, 12s. net; Fasc. II, 12s. net. Supplement III (1901-1905), 28s. net.

Annals of Botany. Edited by I. BAYLEY BALFOUR, D. H. SCOTT,

J. B. FARMER, and R. THAXTER; assisted by other Botanists. Royal 8vo, morocco back, with many plates and illustrations in the text. Subscription price for each four parts, 30s., with 1s. 6d. for foreign postage.

Vol. I, Nos. 1-4. Sold only as part of a complete set.

Vol. II, Nos. 5-8. Sold only as part of a complete set.

Vol. III, Nos. 9-12. £2 12s. 6d. Vol. IV, Nos. 13-16. £2 5s.

Vol. V, Nos. 17-20. £2 10s. Vol. VI, Nos. 21-24. £2 4s.

Vol. VII, Nos. 25-28. £2 10s.

Vol. VIII, Nos. 29-32. £2 10s.

Vol. IX, Nos. 33-36. £2 15s.

Vol. X, Nos. 37-40. Vol. XI, Nos. 41-44.

Vol. XII, Nos. 45-48. Vol. XIII, Nos. 49-52.

Vol. XIV, Nos. 53-56. Vol. XV, Nos. 57-60.

Vol. XVI, Nos. 61-64, including a Life of Sir WILLIAM HOOKER, with a photogravure portrait.

Vol. XVII, Nos. 65-68. Vol. XVIII, Nos. 69-72.

Vol. XIX, Nos. 73-76. Vol. XX, Nos. 77-80.

Vol. XXI, Nos. 81-84. Vol. XXII, Nos. 85-88.

Vol. XXIII, Nos. 89-92. Vol. XXIV, Nos. 93-96.

(The price of a volume is £2 16s. unless stated.)

Indexes to the Annals of Botany. Prepared by T. G. HILL, under

the direction of the Editors. Royal 8vo. (1) Vols. I-X. (2) Vols. XI-XX. Each, paper covers, 5s. to subscribers; to non-subscribers, 9s.; morocco back, to subscribers, 6s.; to non-subscribers, 10s. 6d.

Reprint: Life of Sir WILLIAM HOOKER, with portrait, 3s. 6d. net.

The Flora of Berkshire; with short Biographies of the Berkshire

Botanists. By G. C. DRUCE. Crown 8vo. 16s. net.

The Herbarium of the University of Oxford. By the same.

Crown 8vo. 6d.

The Dillienian Herbaria. By G. C. DRUCE. Edited with Introduction

by S. H. VINES. Crown 8vo. 12s. 6d. net.

Studies in Forestry. By J. NISBET. Crown 8vo. 6s. net.

Forest Terminology (French-German-English). By W. R. FISHER.

[In the press.]

The Physical Properties of Soil. By R. WARINGTON. 8vo. 6s.

School Gardening. By P. ELFORD and S. HEATON. Crown 8vo, illustrated. 2s. net.

A School Garden Note-Book. By PERCY ELFORD and S. HEATON, Crown 8vo. 9d. net.

NATURAL SCIENCE

Biology

- Essays on Evolution.** By E. B. POULTON. 8vo. 12s. net.
- Physiological Histology.** By G. MANN. 8vo. 15s. net.
- Experimental Embryology.** By J. W. JENKINSON. 8vo. 12s. 6d. net.
- Adler's Alternating Generations ; a Biological Study of Oak Galls and Gall Flies.** Translated and edited by C. R. STRATON. With coloured illustrations of 42 Species. Crown 8vo. 10s. 6d. net.
- The Birds of Oxfordshire.** By O. V. AFLIN. 8vo. 7s. 6d. net.
- The Harlequin Fly : its Structure and Life-History.** By L. C. MIALL and A. R. HAMMOND. With one hundred and thirty illustrations. 8vo. 7s. 6d.
- A Naturalist in Tasmania.** By G. W. SMITH. 8vo, illustrated. 7s. 6d. net.
- Müller's Vocal Organs of the Passeres,** translated by F. J. BELL, and edited by A. H. GARROD. With plates. 4to. 7s. 6d.
- A Glossary of Greek Birds.** By D'ARCY W. THOMPSON. 8vo. 10s. net.
- The Order Oligochaeta, a Monograph, Structural and Systematic.** By F. E. BEDDARD. With illustrations. Demy 4to. £2 2s. net.
- Memoirs on the Physiology of Nerve, of Muscle, and of the Electrical Organ.** Edited by Sir J. BURDON-SANDERSON. Medium 8vo. £1 1s.
- Ecker's Anatomy of the Frog.** Translated, with Additions, by G. HASLAM. Medium 8vo. £1 1s.
- Weismann's Essays upon Heredity.** Authorized Translation. Crown 8vo. Vol. I. Ed. E. B. POULTON, S. SCHÖNLAND, and A. E. SHIPLEY. Ed. 2. 7s. 6d. Vol. II. Ed. E. B. POULTON and A. E. SHIPLEY. 5s.
- Catalogue of Eastern and Australian Lepidoptera Heterocera in the Oxford University Museum.** By Colonel C. SWINHOE. Part I : 8vo, with eight plates, £1 1s. Part II : by Col. C. SWINHOE, Lord WALSLINGHAM and J. H. DURRANT. With eight plates, £2 2s.
- African Mimetic Butterflies.** By H. ELTRINGHAM. With ten coloured plates and a map. Royal 4to. £2 10s. net.
- Forms of Animal Life, a manual of comparative Anatomy.** By G. ROLLESTON. Second edition. Medium 8vo. £1 16s.
- Rolleston's Scientific Papers and Addresses.** Edited by W. TURNER. With biographical sketch by E. B. TYLOR. 2 vols. 8vo. £1 4s.
- Nature and Man.** By Sir E. RAY LANKESTER (being the Romanes Lecture, 1905). 8vo, paper covers. Third impression. 2s. net.
- An Introduction to the Study of Biology.** By J. W. KIRKALDY and I. M. DRUMMOND. Crown 8vo, illustrated. 6s. 6d.
-
- Lectures on the Method of Science.** Edited by T. B. STRONG. 8vo. 7s. 6d. net.
- Rivers and Canals.** By L. F. VERNON-HARCOURT. 2 vols. 8vo. 31s. 6d.

Medicine and Hygiene

Quarterly Journal of Medicine. Edited by WILLIAM OSLER, J. ROSE BRADFORD, R. HUTCHISON, A. E. GARROD, H. D. ROLLESTON, and W. HALE WHITE. 8s. 6d. net per number. Annual Subscription, 25s. Vol. I (Nos. 1-4), Vol. II (Nos. 5-8), Vol. III (Nos. 9-12), 35s. net each.

Studies in the Medicine of Ancient India. By A. F. R. HOERNLÉ. 8vo. Part I: Osteology. Illustrated. 10s. 6d.

Surgical Instruments in Greek and Roman Times. By J. S. MILNE. With illustrations. 8vo. 14s. net.

English Medicine in Anglo-Saxon Times: the Fitz-Patrick Lectures for 1903. By J. F. PAYNE. 8vo, with 23 illustrations. 8s. 6d. net.

The Study of Medicine in the British Isles: Fitz-Patrick Lectures, 1905-6. By NORMAN MOORE. 8vo, with 13 Collotype plates. 10s. 6d. net.

The Last Days of Charles II. By RAYMOND CRAWFORD. Medium 8vo, illustrated. 5s. net.

An Alabama Student and other Biographical Essays. By WILLIAM OSLER. 8vo, with portraits and other illustrations. 7s. 6d. net.

Two Oxford Physiologists (Lower and Mayow). By FRANCIS GOTCH. 8vo. 1s. net.

Criminal Responsibility. By CHARLES MERCIER. 8vo. 7s. 6d. net.

The Construction of Healthy Dwellings. By SIR D. GALTON. Second edition. 8vo. 10s. 6d.

Healthy Hospitals. By the same, with illustrations. 8vo. 10s. 6d.

A System of Physical Education: Theoretical and Practical. By A. MACLAREN. New edition, by W. MACLAREN. Crown 8vo. 8s. 6d. net.

Pathological Series in the Oxford Museum. By SIR H. W. ACLAND. 8vo. 2s. 6d.

Epidemic Influenza. By F. A. DIXEY. Medium 8vo. 7s. 6d.

Scarlatina. By D. A. GRESSWELL. Medium 8vo. 10s. 6d.

Surgical Aspect of Traumatic Insanity. By H. A. POWELL. 8vo. 2s. 6d.

The Oxford Medical Publications

PUBLISHED BY

HENRY FROWDE AND HODDER & STOUGHTON

(A complete list may be had on application.)

The Collected Writings of the Right Hon. Lord Lister, O.M., F.R.S. In two volumes. Demy 4to. 42s. net.

The History of Medicine. By Prof. MAX NEUBERGER. Translated by ERNEST PLAYFAIR. In 2 volumes. Crown 4to. Vol. I now ready. 25s. net.

Practical Pathology. By G. SIMS WOODHEAD. Fourth edition. Illustrated by over 240 microscopical drawings in colour. Demy 8vo. 31s. 6d. net.

Clinical Pathology in General Practice. By T. J. HORDER. Demy 8vo. 7s. 6d. net.

A System of Medicine. Edited by WILLIAM OSLER, F.R.S., and THOMAS MCCRAE. In seven 8vo volumes of about 900 pages each illustrated. Price to subscribers for the seven volumes 24s. net each volume; to non-subscribers, single volumes, 30s. net.

MEDICINE AND HYGIENE

- A System of Diet and Dietetics.** From the scientific and practical standpoint. Edited by G. A. SUTHERLAND. Crown 4to. 30s. net.
- A System of Operative Surgery.** Edited by F. F. BURGHARD. Now ready, complete in four vols. Cr. 4to. £6 net the set, or 36s. net each vol.
- Fevers in the Tropics.** By LEONARD ROGERS, I.M.S. Cr. 4to. Ed. 2. 30s. n.
- Heart Diseases.** By JAMES MACKENZIE. 2nd ed. Cr. 4to. 25s. net.
- Heart Diseases—Graphic Methods.** By JOHN HAY. 7s. 6d. net.
- Text-book of Anatomy.** Edited by D. J. CUNNINGHAM, F.R.S. Third edition. Royal 8vo. 31s. 6d. net.
- Manual of Practical Anatomy.** By D. J. CUNNINGHAM, F.R.S. Fourth edition. In two volumes. Crown 8vo. 10s. 6d. net each.
- Manual of Surgery.** By ALEXIS THOMSON and ALEXANDER MILES. Third edition. Two vols. Crown 8vo. 10s. 6d. net each.
- Manual of Operative Surgery.** By H. J. WARING. Third edition. 521 Illustrations, including 30 in colour. Crown 8vo. 12s. 6d. net.
- Operations of General Practice.** By EDRED M. CORNER and HENRY IRVING PINCHES. Third edition. Demy 8vo. 15s. net.
- Fractures and their Treatment.** By J. HOGARTH PRINGLE. Royal 8vo. 15s. net.
- A Handbook of the Surgery of Children.** By Prof. E. KERMISON. Translated by J. KEOGH MURPHY. 8vo. 20s. net.
- Emergencies of General Practice.** By P. SARGENT and A. E. RUSSELL. Demy 8vo. 15s. net.
- Infectious Diseases.** By C. B. KER. Crown 4to. 20s. net.
- Manual of Bacteriology.** By ROBERT MUIR and JAMES RITCHIE. Fifth edition, thoroughly revised. With nearly 200 illustrations in the text and 20 figures in colour. Crown 8vo. 10s. 6d. net.
- A Laboratory Handbook of Bacteriology.** By R. ABEL. Translated by M. H. GORDON, from the Tenth (German) Edition. Cr. 8vo. 5s. net.
- Functional Nervous Disorders in Childhood.** By LEONARD G. GUTHRIE. Second impression. Crown 8vo. 7s. 6d. net.
- Common Disorders and Diseases of Childhood.** By G. D. STILL. Demy 8vo. Second Impression. 15s. net.
- The Blood in Health and Disease.** By R. J. M. BUCHANAN. Demy 8vo. 12s. 6d. net.
- Sprains and Allied Injuries of Joints.** By R. H. A. WHITELOCKE. Demy 8vo. Second edition. 7s. 6d. net.
- Medical Inspection of Schools.** By A. H. HOGARTH. Cr. 8vo. 6s. n.
- Introduction to Practical Chemistry.** By A. M. KELLAS. Demy 8vo. 3s. 6d. net.
- Practical Inorganic Chemistry.** By A. M. KELLAS. Demy 8vo. 5s. n.
- Outlines of Zoology.** By J. ARTHUR THOMSON. Fifth Ed. Revised and Enlarged. Crown 8vo. 12s. 6d. net.

IN PREPARATION

The Surgery of the Skull and Brain. By L. BATHE RAWLING.

Oxford Medical Manuals

Edited by J. KEOGH MURPHY. Crown 8vo. 5s. net each volume.

Diseases of the Larynx. By H. BARWELL. **Treatment of Disease in Children.** By G. A. SUTHERLAND. **Surgical Emergencies.** By P. SARGENT. **Skin Affections in Childhood.** By H. G. ADAMSON. **Heart Disease.** By F. J. POYNTON. **Practical Anaesthetics.** By H. E. G. BOYLE. **Diseases of the Ear.** By HUNTER TOD. **Diseases of the Nose.** By E. B. WAGGETT. **Diseases of the Eye.** By M. S. MAYOU. **Gunshot Wounds.** By Major C. G. SPENCER. **Infant Feeding.** By J. S. FOWLER. **Aseptic Surgery.** By C. B. LOCKWOOD. **Gallstones, their Complications and Treatment.** By A. W. MAYORSON and P. J. CAMMIDGE. **The Law in General Medical Practice.** By S. ATKINSON. **Glandular Enlargements.** By A. EDMUNDS.

Geology

The Face of the Earth (Das Antlitz der Erde). By EDUARD SUSS. Translated by HERTHA B. C. SOLLAS, under the direction of W. J. SOLLAS; with a preface by Professor SUSS. Royal 8vo. In five volumes.
Vol. I, with 6 maps and 48 illustrations. 25s. net.
Vol. II, with 3 maps and 42 illustrations. 25s. net.
Vol. III (= III i of the German edition), with 7 maps and 23 illustrations. 18s. net.
Vol. IV (= III ii of the German edition), with 55 illustrations. 25s. net.

Fossils of the British Islands, Stratigraphically and Zoologically arranged. Part I, Palaeozoic. By R. ETHERIDGE. 4to. £1 10s.

Martini Lister Historia sive Synopsis Methodica Conchyliorum. Editio Tertia. Recensuit et indice locupletissimo instruxit L. W. DILLWYN. 1823. With numerous plates. 52s. 6d. net.

First Lessons in Modern Geology. By A. H. GREEN, edited by J. F. BLAKE. With 42 illustrations. Crown 8vo. 3s. 6d.

Geology of Oxford and Thames. By J. PHILLIPS. 8vo. £1 1s.

Vesuvius. By the same. Crown 8vo. 10s. 6d.

Geology. By Sir J. PRESTWICH. Royal 8vo. Vol. I. Chemical and Physical. £1 5s. Vol. II. Stratigraphical and Physical. With a geological map of Europe. £1 16s. Geological Map (separately), 5s.

Physics of Earthquake Phenomena. By C. G. KNOTT. 8vo, with numerous diagrams. 14s. net.

ASTRONOMY, CHEMISTRY

Astronomy

Astronomy in the Old Testament. By G. SCHIAPARELLI.
Authorized English translation. Crown 8vo. 3s. 6d. net.

A Handbook of Descriptive Astronomy. By G. F. CHAMBERS. Fourth edition. In three vols. 8vo. 28s. net, or separately, Vol. I. Sun, Planets, and Comets, 15s. net. Vol. II. Instruments and Practical Astronomy, 15s. net. Vol. III. The Starry Heavens. 10s. 6d. net.

The Story of the Comets. By G. F. CHAMBERS. 8vo, with many illustrations. Second Edition. 6s. net.

Bradley's Miscellaneous Works. 4to. 17s. net.

A Cycle of Celestial Objects, observed, reduced, and discussed by
W. H. SMYTH; revised and greatly enlarged by G. F. CHAMBERS. 8vo. 6s. net.

Astronomical Observations made at the University Observatory,
Oxford, under the direction of C. PRITCHARD. Royal 8vo.

No. I. On observations of Saturn's Satellites. Paper covers, 3s. 6d.

No. II. A Photometric determination of the magnitudes of all Stars visible to the naked eye, from the Pole to 10° south of the Equator, 8s. 6d.

Nos. III and IV. Researches in Stellar Parallax by the aid of Photography. Part I, 7s. 6d. Part II, 4s. 6d.

Tables for Facilitating Computation of Star-Constants,

By H. H. TURNER. Second impression. 8vo. 2s. (published by Mr. Frowde).

Halley's Comet. By H. H. TURNER. 8vo. Ed. 2. Illustrated. 1s. net.

Halley's Comet, with notes on Comets in general. By G. F. CHAMBERS.
8vo. Illustrated. 1s. net.

Celestial Ejectamenta: the first Halley Lecture, by HENRY WILDE.
8vo. Illustrated. 1s. net.

Chemistry

Mathematical Crystallography. By H. HILTON. 8vo. 14s. net.

Crystallography. A treatise on the Morphology of Crystals. By
N. STORY-MASKELYNE. Crown 8vo. 12s. 6d.

Fock's Introduction to Chemical Crystallography,
translated by W. J. POPE; preface by N. STORY-MASKELYNE. Cr. 8vo. 5s.

The Molecular Tactics of a Crystal. By LORD KELVIN. 8vo. 3s. 6d.

Organic Chemistry of Nitrogen. By N. V. SIDGWICK. 8vo. 14s. net.

Elementary Chemistry. Progressive Lessons in Experiment and
Theory. By F. R. L. WILSON and G. W. HEDLEY. 8vo, with many diagrams.
Part I. Second edition. 3s. Part II. Second edition. 5s.

Class Book of Chemistry. By W. W. FISHER. Ed. 5. Cr. 8vo. 4s. 6d.

Exercises in Practical Chemistry. By A. G. VERNON HARCOURT
and H. G. MADAN. Fifth edition. Crown 8vo. 10s. 6d.

Van 't Hoff's Chemistry in Space, translated and edited by J. E.
MARSH. Crown 8vo. 4s. 6d.

Original Papers in the Science of Chemistry. A list compiled by V. H. VELEY. Third edition. Paper covers, 1s.

Chemistry for Students. By A. W. WILLIAMSON. Fcap. 8vo. 8s. 6d.

Tables of Qualitative Analysis. By H. G. MADAN. 4to. 4s. 6d.

Pure and Applied Mathematics

Introduction to the Algebra of Quantics. By E. B. ELLIOTT. 8vo. 15s. net.

Theory of Continuous Groups. By J. E. CAMPBELL. 8vo. 14s. net.

Theory of Groups of Finite Order. By HAROLD HILTON. 8vo. 14s. net.

Treatise on Infinitesimal Calculus. By BARTHOLOMEW PRICE.

Vol. I. Differential Calculus. [Out of print.]

Vol. II. Integral Calculus, Calculus of Variations, etc. [Out of print.]

Vol. III. Statics, including Attractions; Dynamics of a Material Particle. Second edition. 8vo. 5s.

Vol. IV. Dynamics of Material Systems. Second edition. 8vo. 5s.

The Collected Mathematical Papers of H. J. Stephen Smith, late Savilian Professor of Geometry in the University of Oxford. Edited by J. W. L. GLAISHER. 2 vols. 4to. £3 3s.

Rigaud's Correspondence of Scientific Men of the Seventeenth Century. 1841. 2 vols. 8vo. 18s. 6d.

Elementary Books

Responsions Papers in Stated Subjects (exclusive of Books), 1901-1906, with answers to Mathematical Questions and Introduction by C. A. MARCON and F. G. BRABANT. Crown 8vo. 3s. 6d. net.

A School Course of Mathematics. By DAVID MAIR. Crown 8vo. 3s. 6d.

A Study of Mathematical Education, including the teaching of Arithmetic. By BENCHARA BRANFORD. Crown 8vo. 4s. 6d.

School Algebra. By W. E. PATERSON. Crown 8vo. Parts I (Ed. 3) and II (Ed. 2), each, 2s. 6d. (with answers, each 3s.); together, 4s. (with answers, 5s.).

Arithmetic. With or without answers. By R. HARGREAVES. Crown 8vo. 4s. 6d.

Elementary Plane Trigonometry. By R. C. J. NIXON. Crown 8vo. 7s. 6d.

Book-keeping. By Sir R. G. C. HAMILTON and J. BALL. Fcap 8vo. 2s. Ruled exercise book to the above, 1s. 6d.; to preliminary course only, 4d.

Lectures on the Logic of Arithmetic. By M. E. BOOLE. Crown 8vo. 2s.; or interleaved with writing-paper, 3s.

Figures made Easy: a first Arithmetic Book. By LEWIS HENSLEY. Crown 8vo. 6d. Answers, 1s.

The Scholar's Arithmetic. By the same. 2s. 6d. Answers, 1s. 6d.

The Scholar's Algebra. By the same. Crown 8vo. 2s. 6d.

A Text-Book of Algebra: with Answers. By W. S. ALDIS. Cr. 8vo. 7s. 6d.

Elementary Mechanics of Solids and Fluids. By A. L. SELBY. Crown 8vo. 3s. 6d.

GEOMETRY

Geometry

Geometry for Beginners : an easy introduction to Geometry for young learners. By G. M. MINCHIN. Extra fcap 8vo. 1s. 6d.

Elementary Geometry, on the plan recommended by the
Mathematical Association.

Experimental and Theoretical Course of Geometry.

By A. T. WARREN. With Examination Papers set on the new lines. Crown 8vo. With or without answers. Third edition with additions (1905). 2s.

Elementary Modern Geometry. Part I. Experimental and Theoretical. (Ch. I-IV) Triangles and Parallels. By H. G. WILLIS. Crown 8vo. 1905. 2s.

Euclid Revised, containing the essentials of the Elements of Plane Geometry as given by Euclid in his first six books, edited by R. C. J. NIXON. Third edition. Crown 8vo. 6s.

Sold separately as follows :—Book I, 1s. ; Books I, II, 1s. 6d. ; Books I-IV, 3s. ; Books V, VI, 3s. 6d.

Geometry in Space, containing parts of Euclid's Eleventh and Twelfth Books. By R. C. J. NIXON. Second edition. Crown 8vo. 3s. 6d.

Geometrical Exercises from Nixon's Euclid Revised. With Solutions. By ALEXANDER LARMOR. Crown 8vo. 3s. 6d.

The 'Junior' Euclid. By S. W. FINN. Crown 8vo. Books I and II, 1s. 6d. Books III and IV, 2s.

Sequel to Elementary Geometry, with numerous examples. By J. W. RUSSELL. Cr. 8vo. 6s. **Solutions (Key).** Cr. 8vo. 10s. 6d. net. Obtainable only from the Secretary, Clarendon Press, Oxford.

Pure Geometry, an elementary treatise, with numerous examples. By J. W. RUSSELL. Crown 8vo. New and revised edition (1905). 9s. net.

Analytical Geometry, an elementary treatise by W. J. JOHNSTON. Crown 8vo. 6s.

Notes on Analytical Geometry. By A. CLEMENT JONES. Crown 8vo. 6s. net.

Cremona's Elements of Projective Geometry, translated by C. LEUDESDORF. Second edition. 8vo. 12s. 6d.

Cremona's Graphical Statics, being two treatises on the Graphical Calculus and reciprocal figures in Graphical Statics, translated by T. H. BEARE. 8vo. 8s. 6d.

On the Traversing of Geometrical Figures. By J. COOK WILSON. 8vo, with Addendum. 6s. 6d. net.

Non-Euclidean Geometry. By J. L. COOLIDGE. 8vo. 15s. net.